Exercise Round 4

The deadline of this exercise round is **November 24, 2016**. The solutions will be gone through during the exercise session at 14:15-16:00. The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

The exercise markings should be posted to the "Exercise markings for session" forum on the MyCourses web page by 14:15 on the exercise day. Alternatively you can return PDF solutions to the "Mailbox for pre-exercise reports" on the MyCourses page before 14:15 on the exercise day.

Exercise 1 (Milstein's method)

Consider the following scalar SDE:

$$\mathrm{d}x = -c\,x\,\mathrm{d}t + g\,x\,\mathrm{d}\beta, \qquad x(0) = x_0, \tag{1}$$

where a, g and x_0 are positive constants and $\beta(t)$ is a standard Brownian motion.

(a) Check using the Itô formula that the solution to this equation is

$$x(t) = x_0 \exp\left[(-c - g^2/2)\,t + g\,\beta(t)\right] \tag{2}$$

Hint: $\phi(\beta(t), t) = x_0 \exp\left[(-c - g^2/2)\,t + g\,\beta(t)\right]$.

(b) Simulate trajectories form the equation using Milstein's method with parameters $x_0 = 1$, $c = 1/10$, $g = 1/10$, and check that the histogram at $t = 1$ looks the same as obtained by sampling from the above exact solution.

Exercise 2 (Strong and weak approximations)

Consider the following scalar SDE:

$$\mathrm{d}x = \tanh(x)\,\mathrm{d}t + \mathrm{d}\beta, \qquad x(0) = 0, \tag{3}$$

where $\beta(t)$ is a standard Brownian motion. Recall that it has the exact solution

$$p(x,t) = \frac{1}{\sqrt{2\pi\,t}} \cosh(x) \exp\left(-\frac{1}{2}\,t\right) \exp\left(-\frac{1}{2t}\,x^2\right).$$

Use the following numerical methods for simulating from the model:

(a) Simulate 1000 trajectories from the SDE with the order strong 1.5 Itô–Taylor series based method presented in the lecture notes. Compare the histogram to the exact solution at $t = 5$.

(b) Simulate 1000 trajectories from the SDE with the weak order 2.0 Itô–Taylor series based method presented in the lecture notes using both (i) Gaussian increments, (ii) the three-point distributed random variables. Compare the histograms to the exact solution at $t = 5$.

(c) What can you say about the simulated trajectories when $t \in [0, 5]$ for the different methods?

Exercise 3 (Gaussian approximation of SDEs)

(a) Form a Gaussian assumed density approximation to the SDE in Equation (3) in the time interval $t \in [0, 5]$, and compare it to the exact solution. Compute the Gaussian integrals numerically on a uniform grid.

(b) Form a Gaussian assumed density approximation to Equation (1) and numerically compare it to the histogram obtained in 1(b).