Exercise Round 3

The deadline of this exercise round is **November 17, 2016**. The solutions will be gone through during the exercise session at 14:15-16:00. The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

The exercise markings should be posted to the "Exercise markings for session" forum on the MyCourses web page by 14:15 on the exercise day. Alternatively you can return PDF solutions to the "Mailbox for pre-exercise reports" on the MyCourses page before 14:15 on the exercise day.

Exercise 1 (Fokker–Planck–Kolmogorov (FKP) equation)

(a) Write down the FKP for

$$\mathrm{d}x = \tanh(x)\,\mathrm{d}t + \mathrm{d}\beta, \qquad x(0) = 0, \tag{1}$$

where $\beta(t)$ is a standard Brownian motion, and check that the following solves it:

$$p(x,t) = \frac{1}{\sqrt{2\pi\,t}} \cosh(x) \exp\left(-\frac{1}{2}\,t\right) \exp\left(-\frac{1}{2t}\,x^2\right).$$

(b) Plot the evolution of the probability density when $t \in [0,5]$.

(c) Simulate 1000 trajectories from the SDE using the Euler–Maruyama method and check visually that the histogram matches the correct density at time $t = 5$.

Exercise 2 (Numerical solution of FPK)

Use a finite-differences method to solve the FPK for the Equation (1). For simplicity, you can select a finite range $x \in [-L, L]$ and use the Diriclet boundary conditions $p(-L,t) = p(L,t) = 0$.

(a) Divide the range to n grid points and let $h = 1/(n+1)$. On the grid, approximate the partial derivatives of $p(x,t)$ via

$$\begin{aligned}
\frac{\partial p(x,t)}{\partial x} &\approx \frac{p(x+h,t) - p(x-h,t)}{2h} \\
\frac{\partial^2 p(x,t)}{\partial x^2} &\approx \frac{p(x+h,t) - 2\,p(x,t) + p(x-h,t)}{h^2}.
\end{aligned} \tag{2}$$

(b) Let $\mathbf{p}(t) = (p(h,t)\ p(2h,t)\ \cdots\ p(nh,t))^\mathsf{T}$ and from the above, form an equation of the form

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}\,\mathbf{p}. \tag{3}$$

(c) Solve the above equation using (i) backward Euler, (ii) by numerical computation of $\exp(\mathbf{F}\,t)$, and (iii) by forward Euler. Check that the results match the solution in the previous exercise.

Exercise 3 (Langevin's physical Brownian motion)

Consider the Langevin model of Brownian motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -c\,\frac{\mathrm{d}x}{\mathrm{d}t} + w, \qquad x(0) = (\mathrm{d}x/\,\mathrm{d}t)(0) = 0, \tag{4}$$

where $c = 6\,\pi\,\eta\,r$ and the white noise $w(t)$ has some spectral density q.

(a) Interpret the above model as an Itô SDE and write it as a two-dimensional state-space form SDE.

(b) Write down the differential equations for the elements of the mean $\mathbf{m}(t)$ and covariance $\mathbf{P}(t)$. Conclude that the mean is zero and find the closed-form solutions for the elements $P_{11}(t)$, $P_{12}(t)$, $P_{21}(t)$, and $P_{22}(t)$ of the covariance matrix $\mathbf{P}(t)$. *Hint:* Start by solving $P_{22}(t)$, then use it to find the solutions for $P_{12}(t) = P_{21}(t)$, and finally solve $P_{11}(t)$.

(c) Find the limiting solution $P_{22}(t)$, when $t \to \infty$ and use the following to determine the diffusion coefficient (spectral density) q:

$$m\ \mathrm{E}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right] = \frac{R\,T}{N}. \tag{5}$$

(d) Plot the solution $P_{11}(t)$ and conclude that it asymptotically approaches a straight line. Compute the asymptotic solution $P_{11}(t)$ when $t \to \infty$, and conclude that it approximately gives Langevin's result.