

Exercise Round 2

The deadline of this exercise round is **November 10, 2016**. The solutions will be gone through during the exercise session at 14:15-16:00. The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

The exercise markings should be posted to the "Exercise markings for session" forum on the MyCourses web page by 14:15 on the exercise day. Alternatively you can return PDF solutions to the "Mailbox for pre-exercise reports" on the MyCourses page before 14:15 on the exercise day.

Exercise 1 (Usage of the Itô formula)

- (a) Compute the Itô differential of

$$\phi(\beta) = t + \exp(\beta),$$

where $\beta(t)$ is a Brownian motion with diffusion constant q .

- (b) Compute the Itô differential of

$$\phi(x) = x^2,$$

where x solves the scalar SDE

$$dx = f(x) dt + \sigma d\beta,$$

σ is a constant, and $\beta(t)$ is a standard Brownian motion ($q = 1$).

- (c) Compute the Itô differential of

$$\phi(\mathbf{x}) = \mathbf{x}^\top \mathbf{x},$$

where

$$d\mathbf{x} = \mathbf{F} \mathbf{x} dt + d\boldsymbol{\beta},$$

where \mathbf{F} is a constant matrix and the joint diffusion matrix of $\boldsymbol{\beta}$ is \mathbf{Q} .

Exercise 2 (Stochastic differential equations)

- (a) Check that

$$x(t) = \exp(\beta(t))$$

solves the SDE

$$dx = \frac{1}{2} x dt + x d\beta,$$

where $\beta(t)$ is a standard Brownian motion ($q = 1$).

- (b) Solve the following SDE by changing the variable to $y = \ln x$:

$$dx = -cx \, d\beta,$$

where $c > 0$ is a constant, and $\beta(t)$ is a standard Brownian motion.

- (c) Convert the following Stratonovich SDE equation into the equivalent Itô SDE:

$$\begin{aligned} dx_1 &= -x_2 \circ d\beta, \\ dx_2 &= x_1 \circ d\beta, \end{aligned}$$

where $\beta(t)$ is a scalar Brownian motion.

Exercise 3. (Mean and variance of differential equations)

Derive the mean and covariance equations for the scalar SDE

$$dx = f(x) \, dt + \sigma(x) \, d\beta, \tag{1}$$

where β has the diffusion coefficient q , as follows:

- Take expectations from both sides of the SDE (1) and formally divide by dt to get the differential equation for the mean $m(t)$. Note that the noise term disappears, because $E[\sigma(x(t)) \, d\beta(t)] = 0$.
- Apply the Itô formula to $\phi(x, t) = (x - m(t))^2$ and take the expectation of the resulting equation to derive the differential equation for the variance.
- Write down the mean and covariance differential equations for the scalar SDE

$$dx = -\lambda x \, dt + d\beta,$$

where $\lambda > 0$ and solve them with $x(0) = x_0$.