On Continuous-Discrete Cubature Kalman Filtering

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Abstract: This paper is concerned with application of cubature integration methods to Kalman filtering of discretely observed non-linear stochastic continuous-time systems. We compare two recently proposed variants of the continuous-discrete cubature Kalman filter (CD-CKF), which differ in the order how the discretization and the Gaussian approximation are done. Aside with theoretical analysis we test the performance of the different variants in a simulated application. The results indicate that the relative advantages of the approaches are application specific and neither one is unconditionally better than the other.

Keywords: Kalman filters; nonlinear filters; optimal filtering; stochastic systems; continuous time filters; target tracking.

1. INTRODUCTION

Recently, Särkkä [2007] introduced a continuous-discrete version of the unscented Kalman filter (CD-UKF), which is based on computing the formal continuous-time limit of the discrete-time UKF [Julier et al., 2000]. The CD-UKF can be used for state estimation in continuous-discrete non-linear filtering models [Jazwinski, 1970, Maybeck, 1982] of the form

$$\begin{aligned} dx(t) &= f(x(t), t)\, dt + \sqrt{Q}\, d\beta(t) \\ z_k &= h(x(t_k)) + v_k, \end{aligned} \tag{1}$$

where $x(t)$ is the n-dimensional state, $\beta(t)$ is an n-dimensional standard Brownian motion, and v_k is a zero mean Gaussian random vector with covariance R_k. Because the recently introduced cubature Kalman filter (CKF) [Arasaratnam and Haykin, 2009] can be obtained as a special case of the discrete-time UKF with suitable selection parameters in the unscented transform [cf. Wu et al., 2006, Arasaratnam and Haykin, 2009], the CD-UKF of Särkkä [2007] also includes the continuous-discrete cubature Kalman filter (CD-CKF) as a special case.

More recently, Arasaratnam et al. [2010] introduced a continuous-discrete version of the cubature Kalman filter (CKF) [Arasaratnam and Haykin, 2009], which is applicable to the above class of models, but is based on discretization of the continuous dynamics with Itô-Taylor expansion of strong order 1.5 and then applying the discrete-time CKF. The resulting continuous-discrete cubature Kalman filter (CD-CKF) is different from what we get as the special case of the CD-UKF [Särkkä, 2007], and here we shall analyze these differences in detail and the relative merits of the different formulations.

The Itô-Taylor expansion based method of Arasaratnam et al. [2010] can be considered as a special case of so called *linearized discretization* approach to continuous-discrete filtering [cf. Gustafsson and Isaksson, 1996]. In that method we first discretize the continuous-time dynamic model and then apply discrete-time filter to the resulting discrete time system. The alternative approach is *discretized linearization*, where we first form the filter to the continuous system and then discretize the continuous filter. The latter approach is compatible with the classical framework of continuous-discrete Gaussian filtering [Maybeck, 1982] as well as with the classical continuous-discrete extended Kalman filter (CD-EKF) [Jazwinski, 1970] and recently proposed continuous-discrete unscented Kalman filter (CD-UKF) [Särkkä, 2007]. Cubature integration based filters and smoothers using the latter approach was also recently proposed in [Särkkä and Sarmavuori, 2012].

In this article we present the discretized linearization based continuous-discrete CKF and analyze its advantages and disadvantages over the linearized discretization version. We also show how the new filter can be implemented in square root form and compare its performance to the CD-CKF of Arasaratnam et al. [2010], to other Itô-Taylor expansion based Gaussian filters as well as to CD-UKF of Särkkä [2007] and CD-GHKF of Singer [2008].

1.1 Cubature Integration

In the present context cubature integration refers to methods for approximate computation of Gaussian integrals of the form

$$\mathrm{E}[f(x)] = \int_{\mathbb{R}^n} f(x)\, \mathrm{N}(x \,|\, m, P)\, dx, \tag{2}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^d$ and $\mathrm{N}(x \,|\, m, P)$ is the multidimensional Gaussian density with mean m and covariance matrix P. In particular, cubature integration methods here primarily refer to multidimensional generalizations of

Gauss-Hermite quadrature, that is, to approximations of the form

$$\mathrm{E}[f(x)] \approx \sum_i w_i\, f(x_i), \tag{3}$$

where the weights w_i and the sigma points x_i are functions of the mean m and covariance P of the Gaussian term. In the 3rd order spherical cubature method [see, e.g. Arasaratnam et al., 2010, Arasaratnam and Haykin, 2009], which we mainly discuss here, the sigma points are selected as follows:

$$x_i = m + \sqrt{P}\,\xi_i, \tag{4}$$

where $P = \sqrt{P}\,\sqrt{P}^T$ and the $2n$ unit sigma points ξ_i are selected to be at the intersections of an n-dimensional sphere and the coordinate axes:

$$\xi_i = \begin{cases} \sqrt{n}\, e_i & ,\ i = 1, \ldots, n \\ -\sqrt{n}\, e_{i-n} & ,\ i = n+1, \ldots, 2n. \end{cases} \tag{5}$$

Here e_i denotes a unit vector to the direction of coordinate axis i, and the weights are defined as $w_i = 1/(2n)$ for $i = 1, \ldots, 2n$. The advantage of this method is that the number of evaluation points is linear function of the state dimension. This rule is also a special case of the unscented transform (UT) with a suitable selection of parameters [cf. Wu et al., 2006].

1.2 Discrete-Time Gaussian Filtering and CKF

In discrete-time Gaussian filtering or Gaussian assumed density filtering, one is considered with forming Gaussian approximations to the filtering distributions of models of the following form:

$$\begin{aligned} x_k &= g(x_{k-1}) + w_{k-1} \\ z_k &= h(x_k) + v_k, \end{aligned} \tag{6}$$

where $x_k \in \mathbb{R}^n$ is the discrete-time state sequence, and $w_k \sim \mathrm{N}(0, Q_k^d)$ is the discrete-time process noise sequence.

The *discrete-time Gaussian filter* [Wu et al., 2006, Ito and Xiong, 2000] is the following:

- *Prediction:* Compute predicted state mean and covariance as follows:

$$\begin{aligned} m_k^- &= \mathrm{E}[g(x_{k-1})] \\ P_k^- &= \mathrm{E}[(g(x_{k-1}) - m_k^-)\,(g(x_{k-1}) - m_k^-)^T] + Q_{k-1}^d, \end{aligned} \tag{7}$$

  where all the expectations are taken with respect to the distribution $x_{k-1} \sim \mathrm{N}(m_{k-1}, P_{k-1})$.
- *Update:* Compute the mean and covariance of the state distribution given the measurement z_k:

$$\begin{aligned} \mu_k &= \mathrm{E}[h(x_k)] \\ S_k &= \mathrm{E}[(h(x_k) - \mu_k)\,(h(x_k) - \mu_k)^T] + R_k \\ C_k &= \mathrm{E}[(x_k - m_k^-)\,(h(x_k) - \mu_k)^T] \\ K_k &= C_k\, S_k^{-1} \\ m_k &= m_k^- + K_k\,(z_k - \mu_k) \\ P_k &= P_k^- - K_k\, S_k\, K_k^T, \end{aligned} \tag{8}$$

  where all the expectations are taken with respect to the distribution $x_k \sim \mathrm{N}(m_k^-, P_k^-)$.

The discrete-time CKF [Arasaratnam and Haykin, 2009] is a special case of the above algorithm, where the Gaussian integrals are approximated with 3rd order spherically symmetric cubature integration rule. With different selections of integration methods, we get various other filtering methods [Wu et al., 2006]. The discrete-time UKF [Julier et al., 2000] can also be considered as an approximation to the Gaussian filter.

2. TWO FORMULATIONS OF CD-CKF

2.1 IT-1.5 Based CD-CKF

The method proposed by Arasaratnam et al. [2010] uses the 1.5 order strong Itô-Taylor scheme [Kloeden and Platen, 1999] based discretization of the SDE in model (1) as follows:

$$\begin{aligned} x(t+\delta) = x(t) &+ \delta\, f(x(t), t) + \frac{\delta^2}{2} L_0\, f(x(t), t) \\ &+ \sqrt{Q}\, w + (L\, f(x(t), t))\, y, \end{aligned} \tag{9}$$

where w and y are a suitably correlated pair of zero mean Gaussian random variables and the operators L_0 and L are defined as

$$\begin{aligned} L_0 &= \frac{\partial}{\partial t} + \sum_i f_i\, \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{p,q,j} \sqrt{Q}_{pj} \sqrt{Q}_{qj}\, \frac{\partial^2}{\partial x_p \partial x_q} \\ L &= \sum_{i,j} \sqrt{Q}_{ij}\, \frac{\partial}{\partial x_i}\, e_j^T. \end{aligned} \tag{10}$$

If we define

$$f_d(x(t)) = x(t) + \delta\, f(x(t), t) + \frac{\delta^2}{2} L_0\, f(x(t), t), \tag{11}$$

then the mean $m(t+\delta)$ and covariance $P(t+\delta)$ of $x(t+\delta)$ in approximation (9) can be computed as follows [Arasaratnam et al., 2010]:

$$\begin{aligned} m(t+\delta) &= \mathrm{E}[f_d(x(t))] \\ P(t+\delta) &= \mathrm{E}[f_d(x(t))\, f_d^T(x(t))] - m(t+\delta)\, m^T(t+\delta) \\ &\quad + \frac{\delta^3}{3} \mathrm{E}[(L\, f(x(t), t))(L\, f(x(t), t))^T] \\ &\quad + \frac{\delta^2}{2} \sqrt{Q}\, \mathrm{E}[(L\, f(x(t), t))^T] \\ &\quad + \frac{\delta^2}{2} \mathrm{E}[(L\, f(x(t), t))]\, \sqrt{Q}^T + \delta\, Q, \end{aligned} \tag{12}$$

where the expected values are computed over the distribution $\mathrm{N}(x(t)\,|\,m(t), P(t))$. It is also possible to utilize additional approximations to simplify the equations without significantly affecting the performance [Arasaratnam et al., 2010], but here we use the full version of the algorithm.

In the CD-CKF of Arasaratnam et al. [2010] the primary idea is to approximate the expectations in the above equations with the 3rd order cubature integration method. The algorithm is the following:

- *Prediction*: Divide sampling interval T into M steps of length $\delta = T/M$ iterate the mean and covariance Equations (12) M times while approximating the expectation integrals with 3rd order symmetric cubature integration.
- *Update*: Perform normal discrete-time CKF update step [Arasaratnam and Haykin, 2009], that is, the Gaussian filter update step (8) with 3rd order cubature integration.

Obviously, we could use any other numerical integration scheme [Kloeden and Platen, 1999] for the SDE instead of the Itô-Taylor expansions. The 3rd order cubature integration could also be replaced with some other Gaussian integration method [Wu et al., 2006].

2.2 Classical Gaussian Filtering Based CD-CKF

In classical Gaussian filtering [Maybeck, 1982] and thus also in the CD-UKF of Särkkä [2007] the idea is to approximate the mean and covariance prediction directly in continuous time without discretizing the dynamic model first. The differential equations for the mean and covariance can be derived using the *Itô formula* [see, e.g., Øksendal, 2003], which states that if the evolution of the state is governed by the stochastic differential equation

$$dx = f(x,t)\,dt + \sqrt{Q}\,d\beta, \tag{13}$$

then the time evolution of an arbitrary (scalar) function of state $\phi(x)$ is governed by the equation

$$d\phi = \sum_j \frac{\partial \phi}{\partial x_j}\,[f(x,t)\,dt + \sqrt{Q}\,d\beta]_j + \frac{1}{2}\sum_{ij} Q_{ij}\,\frac{\partial^2 \phi}{\partial x_i \partial x_j}\,dt. \tag{14}$$

Taking expectations from both sides and formally dividing by dt gives the general differential equation for the mean of ϕ as follows:

$$\mathrm{E}\left[\frac{d\phi}{dt}\right] = \mathrm{E}\left[\sum_j \frac{\partial \phi}{\partial x_j}\,f_j(x,t)\right] + \frac{1}{2}\mathrm{E}\left[\sum_{ij} Q_{ij}\,\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right]. \tag{15}$$

By first selecting $\phi(x) = x_i$, and then $\phi(x) = (x_i - m_i)\,(x_j - m_j)$, we get the following results [Jazwinski, 1970]:

$$\begin{aligned}\frac{dm}{dt} &= \mathrm{E}[f(x,t)]\\ \frac{dP}{dt} &= \mathrm{E}[(x-m)\,f^T(x,t)] + \mathrm{E}[f(x,t)\,(x-m)^T] + Q,\end{aligned} \tag{16}$$

where $m = E[x]$ and $P = \mathrm{E}[(x-m)\,(x-m)^T]$. Note that in the Equations (16) the expectations are taken with respect to the actual distribution of the state $p(x(t))$. That is, the equations are not true differential equations in the sense that they cannot be solved without first solving the full Fokker-Planck-Kolmogorov partial differential equation [Jazwinski, 1970].

We shall now use the Gaussian filter [Maybeck, 1982, Ito and Xiong, 2000, Wu et al., 2006] approach, where the idea is to the expectations with respect to the true distribution of $x(t)$ with expectations over the Gaussian approximations. The update step can be approximated in the same way as in the discrete-time Gaussian filter presented in Section 1.2 and thus the *continuous-discrete Gaussian filter* equations can be written as follows:

(1) *Prediction step:* Integrate the mean and covariance differential equations (16) starting from the mean and covariance on the last update time t_{k-1}, to the time t_k of the measurement y_k. The expectations are taken with respect to $x(t) \sim \mathrm{N}(m(t), P(t))$. The results of the prediction are denoted as $m(t_k^-)$, $P(t_k^-)$, where the minus at superscript means "infinite-decimally before the time t_k".

(2) *Update step:* Update step is the same as the discrete-time filter update step in Equations (8), but with the definitions $m_k^- \triangleq m(t_k^-), P_k^- \triangleq P(t_k^-)$ and $m_k \triangleq m(t_k), P_k \triangleq P(t_k)$.

The third-degree rule based *continuous-discrete cubature Kalman filter (CD-CKF)* can be now obtained by substituting the cubature approximations to the continuous-discrete Gaussian filter equations:

(1) *Prediction step:* Integrate the following differential equations from initial conditions m_{k-1}, P_{k-1} at time t_{k-1} to time t_k

$$\begin{aligned}\frac{dm(t)}{dt} &= \frac{1}{2n}\sum_{i=1}^{2n} f(m(t) + \sqrt{P(t)}\,\xi_i, t)\\ \frac{dP(t)}{dt} &= \frac{1}{2n}\sum_{i=1}^{2n} f(m(t) + \sqrt{P(t)}\,\xi_i, t)\,\xi_i^T\,\sqrt{P(t)}^T\\ &\quad + \frac{1}{2n}\sum_{i=1}^{2n} \sqrt{P(t)}\,\xi_i\,f^T(m(t) + \sqrt{P(t)}\,\xi_i, t)\\ &\quad + Q,\end{aligned} \tag{17}$$

where the cubature points ξ_i are defined in Equation (5). The prediction result is $m_k^- = m(t_k)$, $P_k^- = P(t_k)$.

(2) *Update step:* The update step can be obtained by substituting the cubature approximation to the Gaussian filter update Equations (8). First we form the cubature points as

$$x_i = m_k^- + \sqrt{P_k^-}\,\xi_i, \qquad i = 1, \ldots, 2n, \tag{18}$$

and then we approximate the first 3 equations in the Gaussian filter update step (8) as follows:

$$\begin{aligned}\mu_k &= \frac{1}{2n}\sum_{i=1}^{2n} h(x_i)\\ S_k &= \frac{1}{2n}\sum_{i=1}^{2n} (h(x_i) - \mu_k)\,(h(x_i) - \mu_k)^T + R_k\\ C_k &= \frac{1}{2n}\sum_{i=1}^{2n} (x_i - m_k^-)\,(h(x_i) - \mu_k)^T.\end{aligned} \tag{19}$$

Other (higher order) cubature rules could be used for approximating the Gaussian integrals on prediction and update steps in an analogous manner. The above algorithm is also a special case of the CD-UKF Algorithm 4.6 in Särkkä [2007] if we select the UT parameters as $\alpha = 1$, $\beta = 0$ and $\kappa = 0$.

If we define matrix $A(t)$ as the lower triangular Cholesky factor of the covariance $A(t) = \mathrm{chol}(P(t))$, then the differential equations of the prediction step can be transformed into square root form in the same way as in Stengel [1994] and Särkkä [2007], and the result is:

$$
\begin{aligned}
\frac{dm(t)}{dt} &= \frac{1}{2n}\sum_{i=1}^{2n} f(m(t) + A(t)\,\xi_i) \\
\frac{dA(t)}{dt} &= A(t)\,\Phi\Big[\frac{1}{2n}\sum_{i=1}^{2n} A^{-1}(t)\,f(m(t) + A(t)\,\xi_i)\,\xi_i^T \\
&\quad + \frac{1}{2n}\sum_{i=1}^{2n} \xi_i\,f^T(m(t) + A(t)\,\xi_i)\,A^{-T}(t) \\
&\quad + A^{-1}(t)\,Q(t)\,A^{-T}(t)\Big]
\end{aligned} \tag{20}
$$

where $\Phi[\cdot]$ is a function returning the lower triangular part of the argument as follows:

$$
\Phi_{ij}[B] = \begin{cases} B_{ij} \ , \text{ if } i > j \\ \frac{1}{2}B_{ij} \ , \text{ if } i = j \\ 0 \ , \text{ if } i < j. \end{cases} \tag{21}
$$

The update step can be implemented in square root form using the methodology presented in Arasaratnam et al. [2010] and Arasaratnam and Haykin [2009].

2.3 Comparison of Discretizations

The key difference between CD-CKF [Arasaratnam et al., 2010] and CD-CKF presented in Section 2.2 is how the discretization is done:

- In CD-CKF of Arasaratnam et al. [2010] the continuous-time system is first discretized and then approximated as Gaussian. This could be called *linearized discretization* approach.
- In CD-CKF in Section 2.2 the continuous-time system is first approximated with Gaussian process and this Gaussian process is then discretized. This could be called *discretized linearization* approach.

As discussed in Gustafsson and Isaksson [1996], the former approach indeed tends to work better in the case of EKF, as it theoretically should, because in a sense, there are fewer Gaussian approximations involved. In *linearized discretization* the Gaussian approximations are only formed on the discretization steps, but in *discretized linearization* Gaussianity is assumed for all time instants on the sampling interval.

It should also be stressed that unlike claimed in Arasaratnam et al. [2010], the discretized linearization based CD-UKF [Särkkä, 2007] is not based on Euler approximation of the dynamics, but instead, a rigorous application of the Itô formula (see Section 2.2). The reason why we only need to consider terms up to first order in time step size δ, as has been done in Särkkä [2007], is because in the limit $\delta \to 0$ the higher order terms do not contribute anything. That is, we could start from the Itô-Taylor expansion in derivation of the CD-UKF [Särkkä, 2007] and the limit $\delta \to 0$ would still be the same. In fact, it is not hard to see that by taking the formal limit $\delta \to 0$ of Equations (12), we indeed get the Equations (16).

2.4 Order of Convergence in Itô-Taylor

In the Gaussian filtering context, it is not actually the strong order of convergence, which is important, but the weak order of convergence, because one is interested in the moments of the distribution instead of functionals of the path. In the present additive noise case the IT-1.5 actually coincides with the order 2.0 weak Itô-Taylor scheme [Kloeden and Platen, 1999], which was proposed as a suitable SDE discretization method for target tracking applications by Morelande and Gordon [2005]. The model considered by the Morelande and Gordon [2005] was a 2d coordinated turn model with radar measurements, which is a 2d version of the model considered in Arasaratnam et al. [2010].

Although according to Morelande and Gordon [2005] and Arasaratnam et al. [2010], the Itô-Taylor expansion is a suitable choice when the numerical accuracy and computational complexity are concerned, the accuracy does not come without a price. From the definitions of the operators a clear disadvantage of the method can be seen – the discretization depends on the first and second derivatives of the function $x \mapsto f(x,t)$. That is, the derivatives have to exist and analytical forms need to be available. Due to these requirements, one of the key advantages, the derivative-freeness property of the cubature Kalman filtering [Arasaratnam and Haykin, 2009] is lost. However, the need for the closed form derivatives could be eliminated by replacing the Itô-Taylor based method, for example, with one of the Runge-Kutta methods presented in Kloeden and Platen [1999].

3. NUMERICAL COMPARISON

For numerically evaluating the differences between the two filter formulations, we used the same simulation scenario as was used in Arasaratnam et al. [2010]. In this difficult tracking scenario, the state consists of 3d-position, 3d-velocity and the angular velocity of a maneuvering aircraft. We used the same model parameters and repeated the cases with sampling periods $T = 4s$ and $T = 8s$, and angular velocity $\omega = 6^\circ/s$. A difference in our simulation was that instead of using the initialization procedure presented in Section 5.5.3 of Bar-Shalom et al. [2001], we used the initialization presented in the Section 5.5.2. That is, we simply draw the initial estimate from the prior distribution with standard deviation $100m$ in position and velocity components, and $1^\circ/s$ in the angular velocity. We tested the methods in terms of number of divergences (an $> 1000m$ error in position was also considered as a divergence) and root mean squared errors (RMSE) of position.

We tested the following methods:

- CD-EKF: The classical first order CD-EKF [Jazwinski, 1970] with 4th order Runge-Kutta (RK) integration.
- CD-CKF1: The IT-1.5 based continuous-discrete CKF of Arasaratnam et al. [2010].
- CD-UKF1: The IT-1.5 based continuous-discrete UKF with the "classical" parameterization $\alpha = 1, \beta = 0, \kappa = -4$ of Julier et al. [2000].
- CD-GHKF1: The IT-1.5 based continuous-discrete GHKF with 3^n-point Gauss-Hermite (GH) rule.
- CD-CKF2: The Gaussian filtering based continuous-discrete CKF presented in Section 2.2 with 4th order RK integration.

Table 1. Number of divergences in 100 Monte Carlo runs in the case $T = 4s$ as function of number of integration steps M.

Method/M	1	2	4	8	16	32	64
CD-EKF	0	0	0	0	0	0	0
CD-CKF1	0	0	0	0	0	0	0
CD-UKF1	0	0	0	0	0	0	0
CD-GHKF1	0	0	0	0	0	0	0
CD-CKF2	100	100	56	6	0	0	0
CD-UKF2	100	100	77	12	0	0	0
CD-GHKF2	100	100	0	0	0	0	0

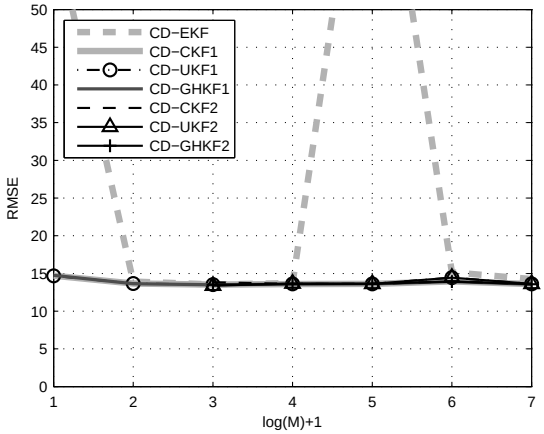

Fig. 1. Position errors as function of number of discretization steps for the case $T = 4s$.

- CD-UKF2: The Gaussian filtering based continuous-discrete UKF [Särkkä, 2007] with 4th order RK-integration.
- CD-GHKF2: The Gaussian filtering based continuous-discrete GHKF [Singer, 2008] with 3^n-point GH rule and 4th order RK-integration.

The filter divergences for the case $T = 4s$ in 100 Monte Carlo simulations are listed in Table 1. Surprisingly, CD-EKF does not diverge in any of the runs and seems to produce meaningful results starting at $M = 1$ discretization steps. The IT-1.5 based CD-CKF1, CD-UKF1 and CD-GHKF1 do not diverge in any of the runs, and they work fine already with only single discretization step. The Gaussian filter based CD-CKF2 and CD-UKF2 diverge in all of the runs when $M = 1$ or $M = 2$, in most of the runs when $M = 4$, and in some of the runs when $M = 8$. Starting at $M = 16$ they do not diverge in any of the runs. The Gaussian filter based CD-GHKF2 diverges in all of the runs when $M = 1$ or $M = 2$, but starts to work reliably from $M = 4$.

The position errors for the case $T = 4s$ as function of the logarithmic number of discretization steps, averaged over the 100 Monte Carlo simulations are shown in Figure 1. It can be seen that the errors of CD-EKF are high with some values of M, and low with some values of M. Thus its performance is very unpredictable and unreliable. All the IT-1.5 based methods CD-CKF1, CD-UKF1 and CD-GHKF1 have practically the same errors. Also the Gaussian filter based methods, in the runs that they do not diverge, have practically the same errors as the IT-1.5 based methods.

Table 2. Number of divergences in 100 Monte Carlo runs in the case $T = 8s$ as function of number of integration steps M.

Method/M	1	2	4	8	16	32	64
CD-EKF	34	0	0	0	0	0	0
CD-CKF1	0	0	0	0	0	0	0
CD-UKF1	23	12	17	14	12	8	19
CD-GHKF1	0	0	0	0	0	0	0
CD-CKF2	100	100	100	5	0	0	0
CD-UKF2	100	100	100	33	14	8	19
CD-GHKF2	100	100	100	0	0	0	0

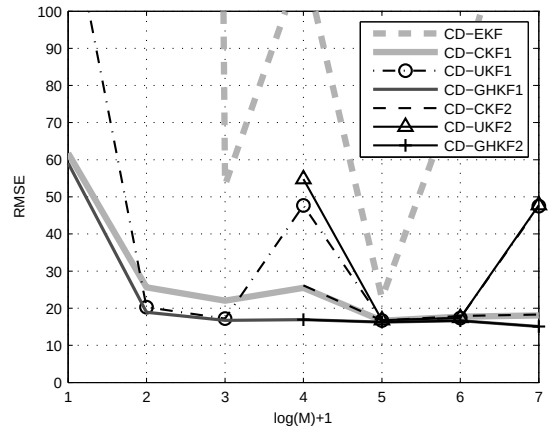

Fig. 2. Position errors as function of number of discretization steps for the case $T = 8s$.

The filter divergences for the case $T = 8s$ are shown in Table 2 and the errors are shown in Figure 2. In this case CD-EKF diverges in some of the runs when the step size is small. However, the errors are an order of magnitude higher than with the other methods and thus, in reality, CD-EKF does not produce meaningful results in this case. Of the IT-1.5 based methods, CD-CKF1 and CD-GHKF1 do not diverge in any of the runs, and they also have the smallest errors with a small number of discretization steps, CD-GHKF1 error being slightly smaller than of CD-CKF1. The IT-1.5 based CD-UKF1 diverges in many of the runs with all of the step sizes. Its error varies a lot with the number of discretization steps, and it sometimes has higher error than CD-CKF1 and sometimes lower.

In the case $T = 8s$ the CD-GHKF2 has slightly less divergences than the other Gaussian filter based CD-CKF2 and CD-UKF2 methods. All these methods start somewhat work at $M = 8$, but CD-UKF2 has divergence problems even with higher number of steps. It is interesting to note that the errors of the corresponding IT-1.5 and Gaussian filter based methods are practically the same after $M = 8$ – as is predicted by the fact that in the limit $\delta \to 0$ the methods converge to the same limit. Starting at $M = 8$ both the Gauss-Hermite based methods CD-GHKF1 and CD-GHKF2 have smaller errors than any of the cubature or unscented transform based methods.

4. CONCLUSION AND DISCUSSION

In this paper, we have analyzed the connection and differences of the continuous-discrete cubature Kalman filter (CD-CKF) obtained as a special of the CD-UKF

[Särkkä, 2007], which amounts to approximating the classical continuous-discrete Gaussian filter equations with cubature integration, and the 1.5 order strong Itô-Taylor (IT-1.5) expansion based CD-CKF presented by Arasaratnam et al. [2010]. The difference in the methods is in the order how the discretization and Gaussian approximation (linearization) are formed. In the limit of infinite number of discretization steps, the algorithms are indeed equivalent, but their numerical performance is different with finite number of discretization steps.

The results indicate that the Gaussian filter based method can have more numerical problems with long discretization step lengths. The explanation for this can be that the IT-1.5 approach has the form that more naturally preserves the positive definiteness of the covariance matrix. When using Runge-Kutta method for the differential equations of mean and covariance, the covariance can more easily become singular or negative definite. It might be possible to avoid this by using some other numerical integration method that would preserve the positive definiteness better.

The usage of the IT-1.5 approach can be beneficial when the discretization steps are long. However, the difference is only present with very small number discretization steps, and when more steps are used, the difference disappears. The IT-1.5 approach has the disadvantage that it requires computation of the first and second order analytical derivatives of the dynamic model function, and thus a single step of the algorithm can be quite computationally heavy. The other disadvantage is that the method cannot be easily extended to case of non-additive noises in dynamic model. The Gaussian filter based approach does not need any analytical derivatives nor their evaluation and it is directly applicable to the non-additive noise case. The relative superiority of the approaches is dependent on the particular application and neither one is always better than the other.

The Gaussian integration method used also has an effect on the performance. The full Gauss-Hermite integration routine works best in both the IT-1.5 and Gaussian filter cases. The second best of the tested ones is the 3rd spherical cubature integration and the third is the unscented transform (with the classical parameters). The most significant difference in the latter two is that the cubature based method is numerically more stable than the unscented transform based method. The explanation for this might be that the classical parameterization of unscented transform causes some of the weights to be negative. Although this choice has some desirable theoretical properties, it easily causes numerical problems. However, one should remember that the cubature rule is just the unscented transform with selection $\alpha = 1$, $\beta = 0$, $\kappa = 0$, and thus we cannot conclude that the cubature method would be better than unscented transform, because both of the methods are indeed unscented transforms.

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