# Lecture 5: Unscented Kalman filter, Gaussian Filter, GHKF and CKF

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# Contents

# Linearization Based Gaussian Approximation

- Problem: Determine the mean and covariance of $y$:

$$x \sim \mathrm{N}(\mu, \sigma^2)$$
$$y = \sin(x)$$

- Linearization based approximation:

$$y = \sin(\mu) + \frac{\partial \sin(\mu)}{\partial \mu}(x - \mu) + \dots$$

which gives

$$\begin{aligned} \mathrm{E}[y] &\approx \mathrm{E}[\sin(\mu) + \cos(\mu)(x - \mu)] = \sin(\mu) \\ \mathrm{Cov}[y] &\approx \mathrm{E}[(\sin(\mu) + \cos(\mu)(x - \mu) - \sin(\mu))^2] = \cos^2(\mu)\, \sigma^2. \end{aligned}$$

# Principle of Unscented Transform [1/3]

- Form 3 sigma points as follows:

$$
\begin{aligned}
\mathcal{X}^{(0)} &= \mu \\
\mathcal{X}^{(1)} &= \mu + \sigma \\
\mathcal{X}^{(2)} &= \mu - \sigma.
\end{aligned}
$$

- We may now select some weights $W^{(0)}, W^{(1)}, W^{(2)}$ such that the original mean and (co)variance can be always recovered by

$$
\begin{aligned}
\mu &= \sum_i W^{(i)} \, \mathcal{X}^{(i)} \\
\sigma^2 &= \sum_i W^{(i)} \, (\mathcal{X}^{(i)} - \mu)^2.
\end{aligned}
$$

# Principle of Unscented Transform [2/3]

- Use the same formula for approximating the distribution of $y = \sin(x)$ as follows:

$$\mu = \sum_i W^{(i)} \sin(\mathcal{X}^{(i)})$$
$$\sigma^2 = \sum_i W^{(i)} (\sin(\mathcal{X}^{(i)}) - \mu)^2.$$

- For vectors $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$ the generalization of standard deviation $\sigma$ is the Cholesky factor $\mathbf{L} = \sqrt{\mathbf{P}}$:

$$\mathbf{P} = \mathbf{L}\,\mathbf{L}^T.$$

- The sigma points can be formed using columns of $\mathbf{L}$ (here $c$ is a suitable positive constant):

$$\mathcal{X}^{(0)} = \mathbf{m}$$
$$\mathcal{X}^{(i)} = \mathbf{m} + c\,\mathbf{L}_i$$
$$\mathcal{X}^{(n+i)} = \mathbf{m} - c\,\mathbf{L}_i$$

# Principle of Unscented Transform [3/3]

- For transformation $\mathbf{y} = \mathbf{g}(\mathbf{x})$ the approximation is:

$$\boldsymbol{\mu}_y = \sum_i W^{(i)}\, \mathbf{g}(\mathcal{X}^{(i)})$$

$$\Sigma_y = \sum_i W^{(i)}\, (\mathbf{g}(\mathcal{X}^{(i)}) - \boldsymbol{\mu}_y)\, (\mathbf{g}(\mathcal{X}^{(i)}) - \boldsymbol{\mu}_y)^T.$$

- Joint distribution of $\mathbf{x}$ and $\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{q}$ is then given as

$$\mathrm{E}\left[\begin{pmatrix} \mathbf{x} \\ \mathbf{g}(\mathbf{x}) + \mathbf{q} \end{pmatrix} \,\middle|\, \mathbf{q}\right] \approx \sum_i W^{(i)} \begin{pmatrix} \mathcal{X}^{(i)} \\ \mathbf{g}(\mathcal{X}^{(i)}) \end{pmatrix} = \begin{pmatrix} \mathbf{m} \\ \boldsymbol{\mu}_y \end{pmatrix}$$

$$\mathrm{Cov}\left[\begin{pmatrix} \mathbf{x} \\ \mathbf{g}(\mathbf{x}) + \mathbf{q} \end{pmatrix} \,\middle|\, \mathbf{q}\right]$$

$$\approx \sum_i W^{(i)} \begin{pmatrix} (\mathcal{X}^{(i)} - \mathbf{m})\, (\mathcal{X}^{(i)} - \mathbf{m})^T & (\mathcal{X}^{(i)} - \mathbf{m})\, (\mathbf{g}(\mathcal{X}^{(i)}) - \\ (\mathbf{g}(\mathcal{X}^{(i)}) - \boldsymbol{\mu}_y)\, (\mathcal{X}^{(i)} - \mathbf{m})^T & (\mathbf{g}(\mathcal{X}^{(i)}) - \boldsymbol{\mu}_y)\, (\mathbf{g}(\mathcal{X}^{(i)})\end{pmatrix}$$

## Unscented transform

The unscented transform approximation to the joint distribution of $\mathbf{x}$ and $\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{q}$ where $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$ and $\mathbf{q} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$ is

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left( \begin{pmatrix} \mathbf{m} \\ \boldsymbol{\mu}_U \end{pmatrix}, \begin{pmatrix} \mathbf{P} & \mathbf{C}_U \\ \mathbf{C}_U^T & \mathbf{S}_U \end{pmatrix} \right),$$

where the sub-matrices are formed as follows:

1. Form the sigma points as

$$\begin{aligned} \mathcal{X}^{(0)} &= \mathbf{m} \\ \mathcal{X}^{(i)} &= \mathbf{m} + \sqrt{n+\lambda} \left[ \sqrt{\mathbf{P}} \right]_i \\ \mathcal{X}^{(i+n)} &= \mathbf{m} - \sqrt{n+\lambda} \left[ \sqrt{\mathbf{P}} \right]_i, \quad i = 1, \ldots, n \end{aligned}$$

## Unscented transform (cont.)

2. Propagate the sigma points through $\mathbf{g}(\cdot)$:

$$\mathcal{Y}^{(i)} = \mathbf{g}(\mathcal{X}^{(i)}), \quad i = 0, \ldots, 2n.$$

3. The sub-matrices are then given as:

$$\begin{aligned}
\boldsymbol{\mu}_U &= \sum_{i=0}^{2n} W_i^{(m)} \, \mathcal{Y}^{(i)} \\
\mathbf{S}_U &= \sum_{i=0}^{2n} W_i^{(c)} \, (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_U) \, (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_U)^T + \mathbf{Q} \\
\mathbf{C}_U &= \sum_{i=0}^{2n} W_i^{(c)} \, (\mathcal{X}^{(i)} - \mathbf{m}) \, (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_U)^T.
\end{aligned}$$

# Unscented Transform [3/3]

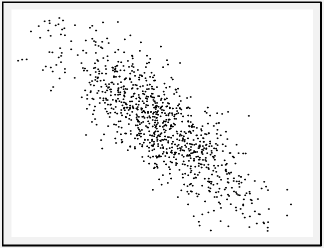
## Unscented transform (cont.)

- $\lambda$ is a scaling parameter defined as $\lambda = \alpha^2 \, (n + \kappa) - n$.
- $\alpha$ and $\kappa$ determine the spread of the sigma points.
- Weights $W_i^{(m)}$ and $W_i^{(c)}$ are given as follows:

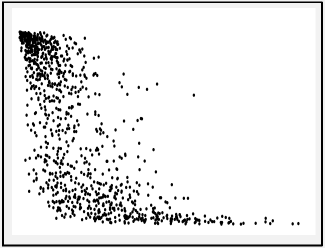
$$
\begin{aligned}
W_0^{(m)} &= \lambda/(n+\lambda) \\
W_0^{(c)} &= \lambda/(n+\lambda) + (1 - \alpha^2 + \beta) \\
W_i^{(m)} &= 1/\{2(n+\lambda)\}, \quad i = 1, \ldots, 2n \\
W_i^{(c)} &= 1/\{2(n+\lambda)\}, \quad i = 1, \ldots, 2n,
\end{aligned}
$$

- $\beta$ can be used for incorporating prior information on the (non-Gaussian) distribution of $\mathbf{x}$.

# Linearization/UT Example



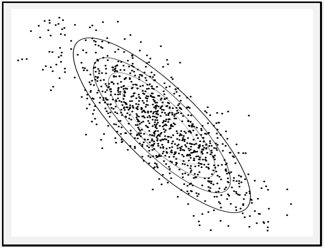
$\Longrightarrow$



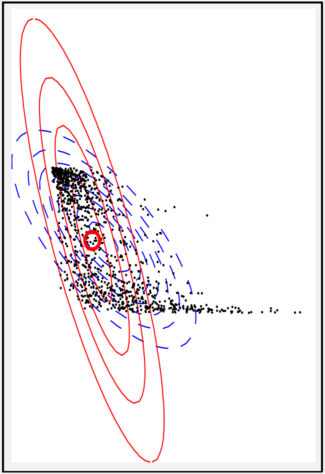
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \mathrm{N}\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} \right)$$

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \exp(-y_1), \quad & y_1(0) &= x_1 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= -\frac{1}{2} y_2^3, & y_2(0) &= x_2 \end{aligned}$$

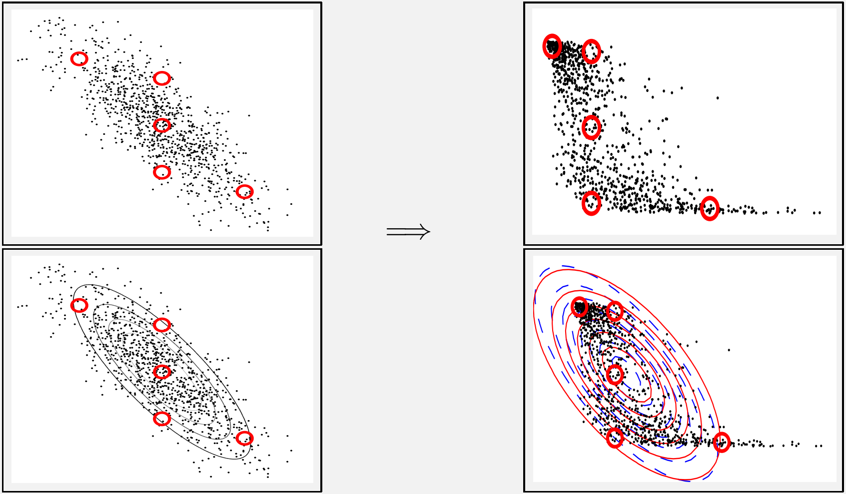
# Linearization Approximation



$\Longrightarrow$



# UT Approximation



# Unscented Kalman Filter (UKF): Derivation [1/4]

- Assume that the filtering distribution of previous step is Gaussian

$$p(\mathbf{x}_{k-1} \mid \mathbf{y}_{1:k-1}) \approx \mathrm{N}(\mathbf{x}_{k-1} \mid \mathbf{m}_{k-1}, \mathbf{P}_{k-1})$$

- The joint distribution of $\mathbf{x}_{k-1}$ and $\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1}$ can be approximated with UT as Gaussian

$$p(\mathbf{x}_{k-1}, \mathbf{x}_k, \mid \mathbf{y}_{1:k-1}) \approx \mathrm{N}\left(\begin{bmatrix}\mathbf{x}_{k-1} \\ \mathbf{x}_k\end{bmatrix} \;\middle|\; \begin{pmatrix}\mathbf{m}'_1 \\ \mathbf{m}'_2\end{pmatrix}, \begin{pmatrix}\mathbf{P}'_{11} & \mathbf{P}'_{12} \\ (\mathbf{P}'_{12})^T & \mathbf{P}'_{22}\end{pmatrix}\right),$$

- Form the sigma points $\mathcal{X}^{(i)}$ of $\mathbf{x}_{k-1} \sim \mathrm{N}(\mathbf{m}_{k-1}, \mathbf{P}_{k-1})$ and compute the transformed sigma points as $\hat{\mathcal{X}}^{(i)} = \mathbf{f}(\mathcal{X}^{(i)})$.
- The expected values can now be expressed as:

$$\mathbf{m}'_1 = \mathbf{m}_{k-1}$$
$$\mathbf{m}'_2 = \sum_i W_i^{(m)} \, \hat{\mathcal{X}}^{(i)}$$

# Unscented Kalman Filter (UKF): Derivation [2/4]

- The blocks of covariance can be expressed as:

$$
\begin{aligned}
\mathbf{P}'_{11} &= \mathbf{P}_{k-1} \\
\mathbf{P}'_{12} &= \sum_i W_i^{(c)} (\mathcal{X}^{(i)} - \mathbf{m}_{k-1}) \, (\hat{\mathcal{X}}^{(i)} - \mathbf{m}'_2)^T \\
\mathbf{P}'_{22} &= \sum_i W_i^{(c)} (\hat{\mathcal{X}}^{(i)} - \mathbf{m}'_2) \, (\hat{\mathcal{X}}^{(i)} - \mathbf{m}'_2)^T + \mathbf{Q}_{k-1}
\end{aligned}
$$

- The prediction mean and covariance of $\mathbf{x}_k$ are then $\mathbf{m}'_2$ and $\mathbf{P}'_{22}$, and thus we get

$$
\begin{aligned}
\mathbf{m}_k^- &= \sum_i W_i^{(m)} \, \hat{\mathcal{X}}^{(i)} \\
\mathbf{P}_k^- &= \sum_i W_i^{(c)} (\hat{\mathcal{X}}^{(i)} - \mathbf{m}_k^-) \, (\hat{\mathcal{X}}^{(i)} - \mathbf{m}_k^-)^T + \mathbf{Q}_{k-1}
\end{aligned}
$$

# Unscented Kalman Filter (UKF): Derivation [3/4]

- For the joint distribution of $\mathbf{x}_k$ and $\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k$ we similarly get

$$p(\mathbf{x}_k, \mathbf{y}_k, \mid \mathbf{y}_{1:k-1}) \approx \mathrm{N}\left( \begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} \;\middle|\; \begin{pmatrix} \mathbf{m}''_1 \\ \mathbf{m}''_2 \end{pmatrix}, \begin{pmatrix} \mathbf{P}''_{11} & \mathbf{P}''_{12} \\ (\mathbf{P}''_{12})^T & \mathbf{P}''_{22} \end{pmatrix} \right),$$

- If $\mathcal{X}^{-(i)}$ are the sigma points of $\mathbf{x}_k \sim \mathrm{N}(\mathbf{m}_k^-, \mathbf{P}_k^-)$ and $\hat{\mathcal{Y}}^{(i)} = \mathbf{h}(\mathcal{X}^{-(i)})$, we get:

$$\begin{aligned}
\mathbf{m}''_1 &= \mathbf{m}_k^- \\
\mathbf{m}''_2 &= \sum_i W_i^{(m)} \hat{\mathcal{Y}}^{(i)} \\
\mathbf{P}''_{11} &= \mathbf{P}_k^- \\
\mathbf{P}''_{12} &= \sum_i W_i^{(c)} (\mathcal{X}^{-(i)} - \mathbf{m}_k^-)\,(\hat{\mathcal{Y}}^{(i)} - \mathbf{m}''_2)^T \\
\mathbf{P}''_{22} &= \sum_i W_i^{(c)} (\hat{\mathcal{Y}}^{(i)} - \mathbf{m}''_2)\,(\hat{\mathcal{Y}}^{(i)} - \mathbf{m}''_2)^T + \mathbf{R}_k
\end{aligned}$$

- Recall that if

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left( \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} \right),$$

then

$$\mathbf{x} \mid \mathbf{y} \sim \mathrm{N}(\mathbf{a} + \mathbf{C}\,\mathbf{B}^{-1}\,(\mathbf{y} - \mathbf{b}), \mathbf{A} - \mathbf{C}\,\mathbf{B}^{-1}\mathbf{C}^T).$$

- Thus we get the conditional mean and covariance:

$$\begin{aligned} \mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{P}''_{12}\,(\mathbf{P}''_{22})^{-1}(\mathbf{y}_k - \mathbf{m}''_2) \\ \mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{P}''_{12}\,(\mathbf{P}''_{22})^{-1}\,(\mathbf{P}''_{12})^T. \end{aligned}$$

# Unscented Kalman Filter (UKF): Algorithm [1/4]

## Unscented Kalman filter: Prediction step

1. Form the sigma points:

$$
\begin{aligned}
\mathcal{X}_{k-1}^{(0)} &= \mathbf{m}_{k-1}, \\
\mathcal{X}_{k-1}^{(i)} &= \mathbf{m}_{k-1} + \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_{k-1}}\right]_i \\
\mathcal{X}_{k-1}^{(i+n)} &= \mathbf{m}_{k-1} - \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_{k-1}}\right]_i, \quad i = 1, \ldots, n.
\end{aligned}
$$

2. Propagate the sigma points through the dynamic model:

$$
\hat{\mathcal{X}}_k^{(i)} = \mathbf{f}(\mathcal{X}_{k-1}^{(i)}). \quad i = 0, \ldots, 2n.
$$

# Unscented Kalman Filter (UKF): Algorithm [2/4]

## Unscented Kalman filter: Prediction step (cont.)

3. Compute the predicted mean and covariance:

$$\mathbf{m}_k^- = \sum_{i=0}^{2n} W_i^{(m)} \, \hat{\mathcal{X}}_k^{(i)}$$

$$\mathbf{P}_k^- = \sum_{i=0}^{2n} W_i^{(c)} \, (\hat{\mathcal{X}}_k^{(i)} - \mathbf{m}_k^-) \, (\hat{\mathcal{X}}_k^{(i)} - \mathbf{m}_k^-)^T + \mathbf{Q}_{k-1}.$$

# Unscented Kalman Filter (UKF): Algorithm [3/4]

## Unscented Kalman filter: Update step

1. Form the sigma points:

$$\mathcal{X}_k^{-(0)} = \mathbf{m}_k^-,$$
$$\mathcal{X}_k^{-(i)} = \mathbf{m}_k^- + \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_k^-}\right]_i$$
$$\mathcal{X}_k^{-(i+n)} = \mathbf{m}_k^- - \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_k^-}\right]_i, \quad i = 1, \ldots, n.$$

2. Propagate sigma points through the measurement model:

$$\hat{\mathcal{Y}}_k^{(i)} = \mathbf{h}(\mathcal{X}_k^{-(i)}), \quad i = 0, \ldots, 2n.$$

# Unscented Kalman Filter (UKF): Algorithm [4/4]

## Unscented Kalman filter: Update step (cont.)

3. Compute the following:

$$
\begin{aligned}
\boldsymbol{\mu}_k &= \sum_{i=0}^{2n} W_i^{(m)}\, \hat{\mathcal{Y}}_k^{(i)} \\
\mathbf{S}_k &= \sum_{i=0}^{2n} W_i^{(c)}\, (\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)\, (\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)^T + \mathbf{R}_k \\
\mathbf{C}_k &= \sum_{i=0}^{2n} W_i^{(c)}\, (\mathcal{X}_k^{-(i)} - \mathbf{m}_k^-)\, (\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)^T \\
\mathbf{K}_k &= \mathbf{C}_k\, \mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k\, [\mathbf{y}_k - \boldsymbol{\mu}_k] \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\, \mathbf{S}_k\, \mathbf{K}_k^T.
\end{aligned}
$$

# Unscented Kalman Filter (UKF): Advantages

- No closed form derivatives or expectations needed.
- Not a local approximation, but based on values on a larger area.
- Functions **f** and **h** do not need to be differentiable.
- Theoretically, captures higher order moments of distribution than linearization.

# Unscented Kalman Filter (UKF): Disadvantage

- Not a truly global approximation, based on a small set of trial points.
- Does not work well with nearly singular covariances, i.e., with nearly deterministic systems.
- Requires more computations than EKF or SLF, e.g., Cholesky factorizations on every step.
- Can only be applied to models driven by Gaussian noises.

# Gaussian Moment Matching [1/2]

- Consider the transformation of $\mathbf{x}$ into $\mathbf{y}$:

$$\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}).$$

- Form Gaussian approximation to $(\mathbf{x}, \mathbf{y})$ by directly approximating the integrals:

$$\mu_M = \int \mathbf{g}(\mathbf{x}) \, \mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P}) \, d\mathbf{x}$$
$$\mathbf{S}_M = \int (\mathbf{g}(\mathbf{x}) - \mu_M) \, (\mathbf{g}(\mathbf{x}) - \mu_M)^T \, \mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P}) \, d\mathbf{x}$$
$$\mathbf{C}_M = \int (\mathbf{x} - \mathbf{m}) \, (\mathbf{g}(\mathbf{x}) - \mu_M)^T \, \mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P}) \, d\mathbf{x}.$$

# Gaussian Moment Matching [2/2]

## Gaussian moment matching

The moment matching based Gaussian approximation to the joint distribution of $\mathbf{x}$ and the transformed random variable $\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{q}$ where $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$ and $\mathbf{q} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$ is given as

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left( \begin{pmatrix} \mathbf{m} \\ \boldsymbol{\mu}_M \end{pmatrix}, \begin{pmatrix} \mathbf{P} & \mathbf{C}_M \\ \mathbf{C}_M^T & \mathbf{S}_M \end{pmatrix} \right),$$

where

$$\begin{aligned} \boldsymbol{\mu}_M &= \int \mathbf{g}(\mathbf{x}) \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x} \\ \mathbf{S}_M &= \int (\mathbf{g}(\mathbf{x}) - \boldsymbol{\mu}_M) \, (\mathbf{g}(\mathbf{x}) - \boldsymbol{\mu}_M)^T \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x} + \mathbf{Q} \\ \mathbf{C}_M &= \int (\mathbf{x} - \mathbf{m}) \, (\mathbf{g}(\mathbf{x}) - \boldsymbol{\mu}_M)^T \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x}. \end{aligned}$$

# Gaussian Filter [1/3]

## Gaussian filter prediction

Compute the following Gaussian integrals:

$$
\begin{aligned}
\mathbf{m}_k^- &= \int \mathbf{f}(\mathbf{x}_{k-1}) \, \mathrm{N}(\mathbf{x}_{k-1} \,|\, \mathbf{m}_{k-1}, \mathbf{P}_{k-1}) \, d\mathbf{x}_{k-1} \\
\mathbf{P}_k^- &= \int (\mathbf{f}(\mathbf{x}_{k-1}) - \mathbf{m}_k^-) \, (\mathbf{f}(\mathbf{x}_{k-1}) - \mathbf{m}_k^-)^T \\
&\qquad \times \mathrm{N}(\mathbf{x}_{k-1} \,|\, \mathbf{m}_{k-1}, \mathbf{P}_{k-1}) \, d\mathbf{x}_{k-1} + \mathbf{Q}_{k-1}.
\end{aligned}
$$

## Gaussian filter update

1. Compute the following Gaussian integrals:

$$
\begin{aligned}
\boldsymbol{\mu}_k &= \int \mathbf{h}(\mathbf{x}_k)\, \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k^-, \mathbf{P}_k^-)\, d\mathbf{x}_k \\
\mathbf{S}_k &= \int (\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)\, (\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)^T\, \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k^-, \mathbf{P}_k^-)\, d\mathbf{x}_k + \mathbf{R}_k \\
\mathbf{C}_k &= \int (\mathbf{x}_k - \mathbf{m}_k^-)\, (\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)^T\, \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k^-, \mathbf{P}_k^-)\, d\mathbf{x}_k.
\end{aligned}
$$

2. Then compute the following:

$$
\begin{aligned}
\mathbf{K}_k &= \mathbf{C}_k\, \mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k\, (\mathbf{y}_k - \boldsymbol{\mu}_k) \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\, \mathbf{S}_k\, \mathbf{K}_k^T.
\end{aligned}
$$

# Gaussian Filter [3/3]

- Special case of assumed density filtering (ADF).
- Multidimensional Gauss-Hermite quadrature ⇒ Gauss Hermite Kalman filter (GHKF).
- Cubature integration ⇒ Cubature Kalman filter (CKF).
- Monte Carlo integration ⇒ Monte Carlo Kalman filter (MCKF).
- Gaussian process / Bayes-Hermite Kalman filter: Form Gaussian process regression model from set of sample points and integrate the approximation.
- Linearization, unscented transform, central differences, divided differences can be considered as special cases.

# Gauss-Hermite Kalman Filter (GHKF) [1/2]

- One-dimensional Gauss-Hermite quadrature of order $p$:

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{N}(x \mid 0, 1)\,dx \approx \sum_{i=1}^{p} W^{(i)} g(x^{(i)}),$$

- $\xi^{(i)}$ are roots of $p$th order Hermite polynomial:

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= x \\
H_2(x) &= x^2 - 1 \\
H_3(x) &= x^3 - 3x \ldots
\end{aligned}$$

- The weights are $W^{(i)} = p!/(p^2\,[H_{p-1}(\xi^{(i)})]^2)$.
- Exact for polynomials up to order $2p-1$.

# Gauss-Hermite Kalman Filter (GHKF) [2/2]

- Multidimensional integrals can be approximated as:

$$
\begin{aligned}
&\int \mathbf{g}(\mathbf{x}) \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x} \\
&= \int \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi) \, \mathrm{N}(\xi \,|\, \mathbf{0}, \mathbf{I}) \, d\xi \\
&= \int \cdots \int \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi) \, \mathrm{N}(\xi_1 \,|\, 0, 1) \, d\xi_1 \times \cdots \times \mathrm{N}(\xi_n \,|\, 0, 1) \, d\xi_n \\
&\approx \sum_{i_1, \ldots, i_n} W^{(i_1)} \times \cdots \times W^{(i_n)} \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi^{(i_1, \ldots, i_n)}).
\end{aligned}
$$

- Needs $p^n$ evaluation points.
- Gauss-Hermite Kalman filter (GHKF) uses this for evaluation of the Gaussian integrals.

# Spherical Cubature Integration [1/3]

- Postulate symmetric integration rule:

$$\int \mathbf{g}(\xi)\,\mathrm{N}(\xi \mid \mathbf{0}, \mathbf{I})\,d\xi \approx W \sum_i \mathbf{g}(c\,\mathbf{u}^{(i)}),$$

where the points $\mathbf{u}^{(i)}$ belong to the symmetric set $[\mathbf{1}]$ with generator $(1, 0, \ldots, 0)$:

$$[\mathbf{1}] = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots \right\}$$

and $W$ is a weight and $c$ is a parameter yet to be determined.

# Spherical Cubature Integration [2/3]

- Due to symmetry, all odd orders integrated exactly.
- We only need to match the following moments:

$$\int \mathrm{N}(\xi \mid \mathbf{0}, \mathbf{I}) \, d\xi = 1$$

$$\int \xi_j^2 \, \mathrm{N}(\xi \mid \mathbf{0}, \mathbf{I}) \, d\xi = 1$$

- Thus we get the equations

$$W \sum_i 1 = W \, 2n = 1$$

$$W \sum_i [c \, u_j^{(i)}]^2 = W \, 2c^2 = 1$$

- Thus the following rule is exact up to third degree:

$$\int \mathbf{g}(\xi) \; \mathrm{N}(\xi \mid \mathbf{0}, \mathbf{I}) \, d\xi \approx \frac{1}{2n} \sum_i \mathbf{g}(\sqrt{n} \, \mathbf{u}^{(i)}).$$

# Spherical Cubature Integration [3/3]

- General Gaussian integral rule:

$$
\begin{aligned}
&\int \mathbf{g}(\mathbf{x}) \; \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x} \\
&= \int \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi) \; \mathrm{N}(\xi \,|\, \mathbf{0}, \mathbf{I}) \, d\xi \\
&\approx \frac{1}{2n} \sum_{i=1}^{2n} \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi^{(i)}),
\end{aligned}
$$

where

$$
\xi^{(i)} = \begin{cases} \sqrt{n} \, \mathbf{e}_i & , \quad i = 1, \ldots, n \\ -\sqrt{n} \, \mathbf{e}_{i-n} & , \quad i = n+1, \ldots, 2n, \end{cases} \tag{1}
$$

where $\mathbf{e}_i$ denotes a unit vector to the direction of coordinate axis $i$.

# Cubature Kalman Filter (CKF) [1/4]

## Cubature Kalman filter: Prediction step

1. Form the sigma points as:

$$\mathcal{X}_{k-1}^{(i)} = \mathbf{m}_{k-1} + \sqrt{\mathbf{P}_{k-1}}\, \xi^{(i)} \qquad i = 1, \ldots, 2n.$$

2. Propagate the sigma points through the dynamic model:

$$\hat{\mathcal{X}}_k^{(i)} = \mathbf{f}(\mathcal{X}_{k-1}^{(i)}). \quad i = 1 \ldots 2n.$$

3. Compute the predicted mean and covariance:

$$\mathbf{m}_k^- = \frac{1}{2n} \sum_{i=1}^{2n} \hat{\mathcal{X}}_k^{(i)}$$

$$\mathbf{P}_k^- = \frac{1}{2n} \sum_{i=1}^{2n} (\hat{\mathcal{X}}_k^{(i)} - \mathbf{m}_k^-)\,(\hat{\mathcal{X}}_k^{(i)} - \mathbf{m}_k^-)^T + \mathbf{Q}_{k-1}.$$

# Cubature Kalman Filter (CKF) [2/4]

## Cubature Kalman filter: Update step

1. Form the sigma points:

$$\mathcal{X}_k^{-(i)} = \mathbf{m}_k^- + \sqrt{\mathbf{P}_k^-}\, \xi^{(i)}, \qquad i = 1, \ldots, 2n.$$

2. Propagate sigma points through the measurement model:

$$\hat{\mathcal{Y}}_k^{(i)} = \mathbf{h}(\mathcal{X}_k^{-(i)}), \quad i = 1 \ldots 2n.$$

# Cubature Kalman Filter (CKF) [3/4]

## Cubature Kalman filter: Update step (cont.)

3. Compute the following:

$$
\begin{aligned}
\boldsymbol{\mu}_k &= \frac{1}{2n} \sum_{i=1}^{2n} \hat{\mathcal{Y}}_k^{(i)} \\
\mathbf{S}_k &= \frac{1}{2n} \sum_{i=1}^{2n} (\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)\,(\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)^T + \mathbf{R}_k \\
\mathbf{C}_k &= \frac{1}{2n} \sum_{i=1}^{2n} (\mathcal{X}_k^{-(i)} - \mathbf{m}_k^-)\,(\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)^T \\
\mathbf{K}_k &= \mathbf{C}_k\,\mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k\,[\mathbf{y}_k - \boldsymbol{\mu}_k] \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\,\mathbf{S}_k\,\mathbf{K}_k^T.
\end{aligned}
$$

# Cubature Kalman Filter (CKF) [4/4]

- Cubature Kalman filter (CKF) is a special case of UKF with $\alpha = 1$, $\beta = 0$, and $\kappa = 0$ – the mean weight becomes zero with these choices.
- Rule is exact for third order polynomials (multinomials) – note that third order Gauss-Hermite is exact for fifth order polynomials.
- UKF was also originally derived using similar way, but is a bit more general.
- Very easy algorithm to implement – quite good choice of parameters for UKF.

# Summary

- Unscented transform (UT) approximates transformations of Gaussian variables by propagating sigma points through the non-linearity.
- In UT the mean and covariance are approximated as linear combination of the sigma points.
- The unscented Kalman filter uses unscented transform for computing the approximate means and covariance in non-linear filtering problems.
- A non-linear transformation can also be approximated with Gaussian moment matching.
- Gaussian filter is based on matching the moments with numerical integration $\Rightarrow$ many kinds of Kalman filters.

# Summary (cont.)

- Gauss-Hermite Kalman filter (GHKF) uses multi-dimensional Gauss-Hermite for approximation of Gaussian filter.
- Cubature Kalman filter (CKF) uses spherical cubature rule for approximation of Gaussian filter – but turns out to be special case of UKF.
- We can also use Gaussian processes, Monte Carlo or other methods for approximating the Gaussian integrals.
- Taylor series, statistical linearization, central differences and many other methods can be seen as approximations to Gaussian filter.

# Unscented/Cubature Kalman Filter (UKF/CKF): Example

- Recall the discretized pendulum model

$$\begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_{k-1}^1 + x_{k-1}^2 \, \Delta t \\ x_{k-1}^2 - g \, \sin(x_{k-1}^1) \, \Delta t \end{pmatrix}}_{\mathbf{f}(\mathbf{x_{k-1}})} + \begin{pmatrix} 0 \\ q_{k-1} \end{pmatrix}$$

$$y_k = \underbrace{\sin(x_k^1)}_{\mathbf{h}(\mathbf{x_k})} + r_k,$$

- $\Rightarrow$ *Matlab demonstration*