Exercise Round 1.

Exercise 1. (Linear Least Squares Estimation)

Assume that we have obtained n measurement pairs (y_k, x_k) from the linear regression model

$$y_k = a_1\, x_k + a_2, \qquad k = 1, \ldots, n. \tag{1}$$

The purpose is now to derive an estimate to the parameters a_1 and a_2 such that the following error is minimized (LS-estimate):

$$E(a_1, a_2) = \sum_{k=1}^{n} (y_k - a_1\, x_k - a_2)^2. \tag{2}$$

A) Define $\mathbf{y} = (y_1 \ \ldots \ y_n)^T$ and $\mathbf{a} = (a_1 \ a_2)^T$. Show that the set of Equations (1) can be written in matrix form

$$\mathbf{y} = \mathbf{X}\, \mathbf{a},$$

with a suitably defined matrix $\mathbf{X}$.

B) Write the error function in Equation (2) in matrix form in terms of $\mathbf{y}$, $\mathbf{X}$ and $\mathbf{a}$.

C) Compute the gradient of the matrix form error function and solve the LS-estimate of the parameter $\mathbf{a}$ by finding the point where the gradient is zero.

Exercise 2. (ZOH Discretization of State Space Model)

Consider the following differential equation model for a forced linear spring:

$$\frac{d^2x(t)}{dt^2} = -c^2\,x(t) + w(t), \qquad x(0) = x_0, x'(0) = v_0,$$

where c, x_0 and v_0 are constants and $w(t)$ is some given force function.

A) Solve the equation as follows:

1. Express the equation in form of first order vector differential equation $d\mathbf{x}/dt = \mathbf{F}\,\mathbf{x} + \mathbf{L}\,w$.
2. Solve the equation by using $\mathbf{G}(t) = \exp(-\mathbf{F}\,t)$ as the integrating factor (note that this is a matrix exponential, not an ordinary one).

B) Assume that $w(t)$ is piecewise constant between the sampling points $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \ldots$ and express the solution in recursive form

$$\mathbf{x}(t_k) = \mathbf{A}\,\mathbf{x}(t_{k-1}) + \mathbf{B}\,w_{k-1},$$

where w_{k-1} is the value of $w(t)$ on the interval $[t_{k-1}, t_k]$.

C) Assume that the piece-wise constant value w_{k-1} is a zero mean Gaussian random variable with variance $q_c/\Delta t$. Further assume that we measure the value $x_1(t_k)$ plus Gaussian noise at the sampling instants. Write the model in form of linear Gaussian state space model.

D) Note that in this exercise we have approximated the stochastic input $w(t)$ as a piece-wise constant process, thus this could be called zeroth-order-hold (ZOH) discretization. However, it is also possible to compute exact discretization for a system, where we model $w(t)$ as a continuous-time white noise process. In that solution the term $\mathbf{B}\,w_{k-1}$ gets replaced with random variable $\mathbf{q}_{k-1}$ with zero mean and the following covariance:

$$\mathbf{Q}_{k-1} = \int_0^{\Delta t} \exp((\Delta t - s)\,\mathbf{F})\,\mathbf{L}\,q_c\,\mathbf{L}^T\,\exp((\Delta t - s)\,\mathbf{F})^T\,ds.$$

Compute this covariance for the case $c = 0$ and compare it to the corresponding ZOH discretization covariance.

Exercise 3. (Kalman filtering with EKF/UKF Toolbox)

A) Download and install the EKF/UKF toolbox to some Matlab computer from the web page:

```
http://www.lce.hut.fi/research/mm/ekfukf/
```

Run the following demonstrations:

```
demos/kf_sine_demo/kf_sine_demo.m
demos/kf_cwpa_demo/kf_cwpa_demo.m
```

After running them read the contents of these files and try to understand how they have been implemented. Also read the documentations of functions kf_predict and kf_update (type e.g. "doc kf_predict" in Matlab).

B) Consider the following state space model:

$$\begin{aligned} \mathbf{x}_k &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}_{k-1} + \mathbf{w}_{k-1} \\ y_k &= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}_k + v_k \end{aligned} \tag{3}$$

where $\mathbf{x}_k = (x_k \;\; \dot{x}_k)^T$ is the state, y_k is the measurement, and $\mathbf{w}_k \sim \mathrm{N}(\mathbf{0}, \mathrm{diag}(1/10^2, 1^2))$ and $v_k \sim \mathrm{N}(0, 10^2)$ are white Gaussian noise processes.

Simulate 100 step state sequence from the model and plot the signal x_k, signal derivative $\dot{x}_k$ and the simulated measurements y_k. Start from initial state drawn from zero-mean 2d-Gaussian distribution with identity covariance.

C) Use Kalman filter for computing the state estimates $\mathbf{m}_k$ using the following kind of Matlab-code:

```
m = [0;0];  % Initial mean
P = eye(2); % Initial covariance
for k = 1:100
   [m,P] = kf_predict(m,P,A,Q);
   [m,P] = kf_update(m,P,y(k),H,R);
   % Store the estimate m of state x_k here
end
```

D) Plot the state estimates $\mathbf{m}_k$, the true states $\mathbf{x}_k$ and measurements y_k. Compute the RMSE error of using the first components of vectors $\mathbf{m}_k$ as the estimates of first components of states $\mathbf{x}_k$. Also compute the RMSE error that we would have if we used the measurements as the estimates.