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Ref.: C. Doran, D. Hestenes, F. Sommen, N. van Acker: *Lie groups as spin groups*, J. Math. Phys., **34** (8) (1993), 3642-3669.

Dear Chris Doran, David Hestenes and Frank Sommen,

On page 3650 line 23 and page 3653 lines 26-27, you claim to have proved, in finite dimensions, that (a conjecture of Hestenes & Sobczyk 1984, page 297)

(*) every Lie algebra is a subalgebra of $so(n,n)$.

Many proofs of (*) were well-known before 1984. The result (*) is a corollary of Ado's theorem, which states that every finite dimensional Lie algebra has a faithful representation in finite dimensions (in characteristic 0). To see this, associate to a matrix M in $gl(n,\mathbb{R})$ the matrix (M^T is the transpose of M)

$$\begin{pmatrix} M & 0 \\ 0 & -M^T \end{pmatrix}$$

in the intersection of $so(n,n)$ and $sp(2n,\mathbb{R})$, where $\mathbb{R}^{2n}$ has two bilinear forms on itself (with $\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}' \in \mathbb{R}^n$), one symmetric and one antisymmetric:

$$((\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}')) \to \mathbf{x}\cdot\mathbf{y}' + \mathbf{x}'\cdot\mathbf{y} \quad \text{and} \quad ((\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}')) \to \mathbf{x}\cdot\mathbf{y}' - \mathbf{x}'\cdot\mathbf{y}.$$

On the group level the corollary implies $GL(n,\mathbb{R}) = O(n,n) \cap Sp(2n,\mathbb{R})$.

You prove the statement (*) with Clifford algebras by an identification between $\mathrm{End}(\mathcal{C}\ell_n)$ and $\mathcal{C}\ell_{n,n}$, which leads to an embedding $GL(n,\mathbb{R}) \subset SO(n,n)$. You regard your proof as original, while matrices are not used. However, $GL(n,\mathbb{R})$ was known to be a subgroup of $SO(n,n)$ without reference to matrices or Clifford algebras. To wit, take a linear space V with dual V^*. Associate to $(f,\mathbf{x})$ in $V^*\times V$ the form $f(\mathbf{x})$. This makes $V^*\times V$ a quadratic space, with isometry group $O(V^*\times V)$. By definition $f(L\mathbf{x}) = (L^t f)(\mathbf{x})$ for L in $\mathrm{End}(V)$. Thus, $L \in GL(V)$ satisfies $(L^t f)(L^{-1}\mathbf{x}) = f(\mathbf{x})$, which means that L is an isometry of $V^*\times V$, that is, $GL(V)$ is included in $O(V^*\times V)$. This well-known proof neither refers to matrices nor invokes the Clifford algebra $\mathcal{C}\ell(V^*\times V) = \mathrm{End}(\bigwedge V)$, which is unnecessary for injecting $GL(V) = GL(n,\mathbb{R})$ into $SO(V^* \times V) = SO(n,n)$.

On page 3653 lines 27-28 the statement every Lie group is isomorphic to a subgroup of a general linear group is false and on page 3642 in the Abstract the statement every Lie group can be represented as a spin group is wrong. Known counterexamples are the universal covering groups $\widetilde{SL(m,\mathbb{R})}$ of $SL(m,\mathbb{R})$, $m>1$, which do not have faithful representations in finite dimensions, that is, which are not subgroups of any $GL(n,\mathbb{R})$. The group $SL(2,\mathbb{R})$ is homeomorphic to $S^1\times\mathbb{R}^2$ and so $\widetilde{SL(2,\mathbb{R})}$ is a $\mathbb{Z}$-fold cover of $SL(2,\mathbb{R})$, while for $m>2$, $\widetilde{SL(m,\mathbb{R})}$ is a two-fold cover of $SL(m,\mathbb{R})$. Thus, $\widetilde{SL(m,\mathbb{R})}$ contains $\mathbf{Spin}(m)$, for $m>2$, and faithful representation spaces of $\widetilde{SL(m,\mathbb{R})}$ contain those of $\mathbf{Spin}(m)$. A finite dimensional representation of the Lie algebra $sl(m,\mathbb{R})$ reduces to simple tensors. Spinors are not tensors, that is, no tensor representation gives a faithful representation of the group $\mathbf{Spin}(m)$. Thus, the universal covering group $\widetilde{SL(m,\mathbb{R})}$ of $SL(m,\mathbb{R})$, $m>2$, has no faithful representations in finite dimensions, that is, it is not a subgroup of any $GL(n,\mathbb{R})$.

On page 3642 the first sentence of the Abstract reads every Lie algebra can be represented as a bivector algebra; hence every Lie group can be represented as a spin group. The conclusion "hence" is faulty: an injective homomorphism $g \to so(n,n)$ between Lie algebras is not in correspondence with a homomorphism $G \to \mathbf{Spin}(n,n)$ between Lie groups. There are two mistakes in "hence":
1) there is a homomorphism $\tilde{G} \to \mathbf{Spin}(n,n)$ defined on the universal cover $\tilde{G}$ (simply connected) of G; one needs to verify that the kernel of $\tilde{G} \to \mathbf{Spin}(n,n)$ contains the kernel of $\tilde{G} \to G$ in order to obtain a factorization traversing G: $\tilde{G} \to G \to \mathbf{Spin}(n,n)$;
2) if a homomorphism $G \to \mathbf{Spin}(n,n)$ exists, one cannot conclude that it is injective; only that its kernel is of dimension 0.

On page 3650 lines 16-18 the formula (4.9) is wrong. Counterexample in $\mathbb{R}^{2,2}$: There are elements in $\mathbf{Spin}_+(2,2)$, which cannot be written in the form $\pm\exp(\mathbf{B})$, $\mathbf{B}$ a bivector; for instance $\pm\mathbf{e}_{1234}\exp(b\mathbf{F})$, $\mathbf{F} = \mathbf{e}_{12} + 2\mathbf{e}_{14} + \mathbf{e}_{34}$, $b>0$. [See M. Riesz: *Clifford numbers and spinors*, 1958/1993, pp. 150-152, 170-171.]

To summarize: In your paper *Lie groups as spin groups* you are mistaken on both of your topics, *Lie groups* and *spin groups*, as well as on your main issue of *representing* Lie groups *as* spin groups.

Sincerely Yours,

Pertti Lounesto

Yet, one of the authors insists correctness on page 19 lines 10-11 of D. Hestenes, H. Li, A. Rockwood: "New algebraic tools for classical geometry", pages 3-26 in G. Sommer: "Geometric Computing with Clifford Algebras, theoretical foundations and applications in computer vision and robotics", Springer, 2001, were they write "It has been proved in (1993) that every Lie group can be represented as a spin group".