

BLIND UNITARY PREWHITENING WITH A REAL-VALUED EIGENDECOMPOSITION*

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Abstract. Prewhitening is a standard step for the processing of noisy signals. Typically, eigenvalue decomposition (EVD) of the sample data covariance matrix is used to calculate the whitening matrix. From a computational point of view, an important problem here is to reduce the complexity of the EVD of the complex-valued sample data covariance matrix. In this paper, we show that the computational complexity of the prewhitening step for complex-valued signals can be reduced approximately by a factor of four when the real-valued EVD is used instead of the complex-valued one. Such complexity reduction can be achieved for any axis-symmetric array. The performance of the proposed procedure is studied in application to a blind source separation (BSS) problem. For this application, the performance of the proposed prewhitening scheme is illustrated by means of simulations, and compared with the conventional prewhitening scheme. Among a number of BSS methods which use prewhitening, the second-order blind identification procedure has been adopted in this paper.

Key words: Prewhitening, unitary transform, centro-Hermitian matrices, blind source separation.

1. Introduction

For the processing of noisy signals, the prewhitening step is typically used before applying a particular signal processing procedure [10]. In terms of computational complexity, the prewhitening step often becomes a bottleneck step. For example, the complexity of many blind source separation (BSS) methods [3] essentially depends on the complexity of the prewhitening step. This is especially true for the processing of complex-valued signals. Therefore, in practice, it is important to reduce the complexity of the prewhitening step when processing noisy complex-

* Received April 27, 2005; revised August 10, 2005.

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valued signals. Reduced complexity of the prewhitening step is also appealing from a hardware implementation point of view.

A technique that enables us to reduce the computational complexity of the matrix inversion or eigenvalue decomposition (EVD) for centro-Hermitian matrices has been published in the mathematical literature [6]. This technique makes use of the result that a complex-valued centro-Hermitian matrix can be uniquely transformed into a real-valued one. Then the computational complexity of the inversion or EVD of the real-valued matrix can be reduced approximately by a factor of four as compared to the inversion or EVD of a complex-valued matrix. The latter fact has been widely exploited in array processing to improve the resolution of direction-of-arrival (DOA) estimation methods [4], [8]. Specifically, a big class of DOA estimation methods, such as MUSIC, root-MUSIC, and ESPRIT, exploits the decomposition of a complex-valued array covariance matrix into signal and noise subspaces. Such decomposition is usually performed using the EVD. The modification of the ESPRIT procedure with reduced complexity that is based on the transformation of the complex-valued covariance matrix into a real-valued one has been proposed in [4]. A similar approach has been taken in [8] to reduce the complexity of the root-MUSIC estimator.

Another fruitful application of the technique of [6] as applied to the EVD, which has not been addressed before, is the prewhitening of complex-valued signals.

In this paper, we propose a simple procedure that enables the reduction of the computational complexity of the prewhitening step for complex-valued signals by a factor of four. The proposed procedure makes use of forward-backward (FB) averaging [9], [11] and the aforementioned transformation of a complex-valued covariance matrix into a real-valued one. Therefore, it can be applied only to axis-symmetrical arrays [7]. However, the latter fact should not be taken as a serious limitation because the most appealing array geometry in practice—the uniform linear array (ULA)—belongs to the class of axis-symmetrical arrays. A specific application to the BSS problem is considered.

2. Data model

We consider an axis-symmetrical array [7] of K sensors receiving the signals from M narrowband sources, $M \leq K$. The $K \times 1$ snapshot vector of antenna array outputs can be written as

$$\mathbf{y}(n) = \mathbf{x}(n) + \mathbf{v}(n) = \mathbf{A}\mathbf{s}(n) + \mathbf{v}(n), \quad (1)$$

where, in fairly common notation, $\mathbf{x}(n) = [x_1(n), \dots, x_K(n)]^T \in \mathbb{C}^{K \times 1}$ is the vector that contains the noiseless array output sampled at time n , $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_M] \in \mathbb{C}^{K \times M}$ is the matrix transfer complex operator between sources and sensors, $\mathbf{a}_m = [a_{1,m}, \dots, a_{K,m}]^T \in \mathbb{C}^{K \times 1}$ is the spatial signature of

the m th source signal, $\mathbf{s}(n) = [s_1(n), \dots, s_M(n)]^T \in \mathbb{C}^{M \times 1}$ is the vector of the source waveforms, $\mathbf{v}(n) = [v_1(n), \dots, v_K(n)]^T \in \mathbb{C}^{K \times 1}$ is the vector of additive spatially and temporally white Gaussian noise, and $[\cdot]^T$ denotes the transpose. We also assume that the sources are spatially uncorrelated to each other and to noise.

The basic problem of array signal processing is to estimate the signals $s_1(n), \dots, s_M(n)$, $n = 1, \dots, N$ using only the array measurements $y_1(n), \dots, y_K(n)$, $n = 1, \dots, N$, where N is the total number of available snapshots. Equivalently the problem can be formulated as an estimation of the matrix \mathbf{A} based only on the array measurements $y_1(n), \dots, y_K(n)$, $n = 1, \dots, N$.

3. Prewhitening

Prewhitening is achieved by applying to $\mathbf{x}(n)$ a whitening matrix \mathbf{W} , i.e., an $M \times K$ matrix verifying

$$E\{\mathbf{W}\mathbf{x}(n)\mathbf{x}^H(n)\mathbf{W}^H\} = \mathbf{W}\mathbf{R}_x(0)\mathbf{W}^H = \mathbf{W}\mathbf{A}\mathbf{A}^H\mathbf{W}^H = \mathbf{I} \quad (2)$$

where

$$\mathbf{R}_x(0) = E\{\mathbf{x}(n)\mathbf{x}^H(n)\} = \mathbf{A}\mathbf{R}_s(0)\mathbf{A}^H = \mathbf{A}\mathbf{A}^H \quad (3)$$

is the noiseless array output covariance matrix, $E\{\cdot\}$ and $[\cdot]^H$ stand for expectation and Hermitian transpose operations, respectively, and \mathbf{I} is the identity matrix. Without any loss of generality, we assume here that the source covariance matrix $\mathbf{R}_s(0) = \mathbf{I}$, i.e., the source signals have unit variance so that the dynamic range of the sources is accounted for by the magnitude of the corresponding columns of \mathbf{A} [1]. Equation (2) shows that if \mathbf{W} is a whitening matrix, then $\mathbf{W}\mathbf{A}$ is an $M \times M$ unitary matrix. Equivalently, for any whitening matrix \mathbf{W} , there exists an $M \times M$ unitary matrix \mathbf{U} such that $\mathbf{W}\mathbf{A} = \mathbf{U}$. Then matrix \mathbf{A} can be factored as

$$\mathbf{A} = \mathbf{W}^\dagger \mathbf{U}, \quad (4)$$

where $[\cdot]^\dagger$ denotes the Moore-Penrose pseudoinverse. The whitened process $\mathbf{z}(n)$ obeys a linear model:

$$\mathbf{z}(n) = \mathbf{W}\mathbf{y}(n) = \mathbf{W}[\mathbf{A}\mathbf{s}(n) + \mathbf{v}(n)] = \mathbf{U}\mathbf{s}(n) + \mathbf{W}\mathbf{v}(n), \quad (5)$$

where the signal part of $\mathbf{z}(n)$ is a ‘‘unitary mixture’’ of the source signals.

Because the noiseless array output $\mathbf{x}(n)$ is not known and $\mathbf{R}_x(0)$ cannot be directly computed, the whitening matrix \mathbf{W} can be determined from the noisy array output covariance matrix

$$\mathbf{R}_y(0) = E\{\mathbf{y}(n)\mathbf{y}^H(n)\} = \mathbf{A}\mathbf{A}^H + \sigma^2\mathbf{I}, \quad (6)$$

provided the noise variance σ^2 (or in general noise covariance matrix) is known or can be estimated. Obviously, that $\mathbf{R}_x(0)$ then can be computed as

$$\mathbf{R}_x(0) = \mathbf{R}_y(0) - \sigma^2\mathbf{I}. \quad (7)$$

Typically, \mathbf{W} is obtained using the EVD of the sample estimate of the covariance matrix (6) calculated as

$$\widehat{\mathbf{R}}_y(0) = \frac{1}{N} \sum_{n=1}^N \mathbf{y}(n) \mathbf{y}^H(n). \quad (8)$$

The computation of the EVD of the complex-valued matrix usually constitutes a significant amount of overall computations required for a specific signal processing algorithm. In this paper, we show that the EVD of the complex-valued sample covariance matrix $\widehat{\mathbf{R}}_y(0)$ can be equivalently substituted by the EVD of a specially constructed real-valued covariance matrix. It leads to significant computational savings, because the real-valued EVD has a reduced computational complexity as compared to the complex one, approximately by a factor of four [5].

4. Unitary prewhitening

The covariance matrix $\mathbf{R}_y(0)$ (6) is centro-Hermitian because the following property [6]:

$$\mathbf{R}_y(0) = \mathbf{J} \mathbf{R}_y^*(0) \mathbf{J} \quad (9)$$

holds. Here $[\cdot]^*$ stands for the complex conjugate, and \mathbf{J} is the exchange matrix:

$$\mathbf{J} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}. \quad (10)$$

However, the sample covariance matrix $\widehat{\mathbf{R}}_y(0)$ (8) is not necessarily centro-Hermitian because of the finite sample size effect. However, the centro-Hermitian property can be forced by FB averaging:

$$\widehat{\mathbf{R}}_{FB} = \frac{1}{2} (\widehat{\mathbf{R}}_y(0) + \mathbf{J} \widehat{\mathbf{R}}_y^*(0) \mathbf{J}). \quad (11)$$

It is well known that by using FB averaging the number of snapshots can be nearly doubled and possibly correlated source pairs can be decorrelated [9], [11]. The latter facts often lead to improvement of the estimation accuracy for the methods that use $\widehat{\mathbf{R}}_{FB}$ as compared to the methods that use $\widehat{\mathbf{R}}_y(0)$. It is especially true if the number of available snapshots is small.

Next we can introduce the real-valued covariance matrix, which uniquely corresponds to $\widehat{\mathbf{R}}_{FB}$, as

$$\widehat{\mathbf{C}} = \mathbf{Q}^H \widehat{\mathbf{R}}_{FB} \mathbf{Q} \quad (12)$$

where \mathbf{Q} is one of the following $K \times K$ sparse unitary matrices [4], [6] matrices:

$$\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I} & j\mathbf{I} \\ \mathbf{J} & -j\mathbf{J} \end{bmatrix}, \quad \mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I} & \mathbf{0} & j\mathbf{I} \\ \mathbf{0}^T & \sqrt{2} & \mathbf{0}^T \\ \mathbf{J} & \mathbf{0} & -j\mathbf{J} \end{bmatrix}, \quad (13)$$

where $\mathbf{0}$ denotes the vector which contains only zeros, and $j = \sqrt{-1}$. The matrix of even dimension can be chosen for an array with an even number of sensors, and the matrix of odd dimension can correspond to an array with an odd number of sensors.

It is interesting to note that the real-valued covariance matrix (12) can also be obtained via the forward-only covariance matrix $\widehat{\mathbf{R}}_y(0)$ as

$$\widehat{\mathbf{C}} = \frac{1}{2} (\mathbf{Q}^H \widehat{\mathbf{R}}_y(0) \mathbf{Q} + \mathbf{Q}^H \mathbf{J} \widehat{\mathbf{R}}_y^*(0) \mathbf{J} \mathbf{Q}) = \text{Re}\{\mathbf{Q}^H \widehat{\mathbf{R}}_y(0) \mathbf{Q}\}, \quad (14)$$

where $\text{Re}\{\cdot\}$ stands for the real part of the complex matrix. The property (14) helps to further reduce the computational complexity by reducing the number of algebraic operations required for calculation of $\widehat{\mathbf{C}}$. Thus, the increase in complexity caused by having to compute (14) for given $\widehat{\mathbf{R}}_y(0)$ is negligible.

In order to estimate the whitening matrix \mathbf{W} , we propose to use the EVD of the real-valued matrix $\widehat{\mathbf{C}}$ instead of the EVD of the complex-valued matrix $\widehat{\mathbf{R}}_y(0)$. This reduces the computational complexity of the prewhitening step approximately by a factor of four [5]. The EVDs of matrices $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{R}}_{FB}$ can be written as

$$\widehat{\mathbf{C}} = \widehat{\mathbf{E}}_s \widehat{\mathbf{\Lambda}}_s \widehat{\mathbf{E}}_s^H + \widehat{\mathbf{E}}_n \widehat{\mathbf{\Lambda}}_n \widehat{\mathbf{E}}_n^H \quad (15)$$

$$\widehat{\mathbf{R}}_{FB} = \widehat{\mathbf{L}}_s \widehat{\mathbf{\Gamma}}_s \widehat{\mathbf{L}}_s^H + \widehat{\mathbf{L}}_n \widehat{\mathbf{\Gamma}}_n \widehat{\mathbf{L}}_n^H, \quad (16)$$

where $\widehat{\mathbf{E}}_s = [\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_M]$, $\widehat{\mathbf{E}}_n = [\hat{\mathbf{e}}_{M+1}, \dots, \hat{\mathbf{e}}_K]$, $\widehat{\mathbf{\Lambda}}_s = \text{diag}(\hat{\lambda}_1, \dots, \hat{\lambda}_M)$, and $\widehat{\mathbf{\Lambda}}_n = \text{diag}(\hat{\lambda}_{M+1}, \dots, \hat{\lambda}_K)$ are the matrices of eigenvectors and eigenvalues, which correspond, respectively, to the signal and noise subspaces of the real-valued covariance matrix $\widehat{\mathbf{C}}$, and $\widehat{\mathbf{L}}_s = [\hat{\mathbf{l}}_1, \dots, \hat{\mathbf{l}}_M]$, $\widehat{\mathbf{L}}_n = [\hat{\mathbf{l}}_{M+1}, \dots, \hat{\mathbf{l}}_K]$, $\widehat{\mathbf{\Gamma}}_s = \text{diag}(\hat{\gamma}_1, \dots, \hat{\gamma}_M)$, and $\widehat{\mathbf{\Gamma}}_n = \text{diag}(\hat{\gamma}_{M+1}, \dots, \hat{\gamma}_K)$ are the matrices of eigenvectors and eigenvalues, which correspond, respectively, to the signal and noise subspaces of the FB covariance matrix $\widehat{\mathbf{R}}_{FB}$.

Using the properties that $\widehat{\mathbf{E}} = \mathbf{Q}^H \widehat{\mathbf{L}}$ and $\widehat{\mathbf{\Lambda}} = \widehat{\mathbf{\Gamma}}$, the complex-valued whitening matrix can be calculated as

$$\widehat{\mathbf{W}} = \widehat{\mathbf{L}}_s (\widehat{\mathbf{\Gamma}}_s - \hat{\sigma}^2 \mathbf{I})^{-1/2} = \mathbf{Q}^{-H} \widehat{\mathbf{E}}_s (\widehat{\mathbf{\Lambda}}_s - \hat{\sigma}^2 \mathbf{I})^{-1/2}, \quad (17)$$

where

$$\hat{\sigma}^2 = \frac{1}{K-M} \sum_{i=M+1}^K \hat{\lambda}_i \quad (18)$$

is the averaging of the $K - M$ smallest eigenvalues of $\widehat{\mathbf{C}}$, which correspond to the noise variance.

Note that in the last expression of (17) only the matrices of eigenvectors and eigenvalues of the real-valued covariance matrix $\widehat{\mathbf{C}}$ are used. Then, the whitened signals can be written as

$$\mathbf{z}(n) = \widehat{\mathbf{W}}\mathbf{y}(n) = \mathbf{Q}^{-H}\widehat{\mathbf{E}}_s(\widehat{\mathbf{\Lambda}}_s - \hat{\sigma}^2\mathbf{I})^{-1/2}\mathbf{y}(n). \quad (19)$$

The proposed blind unitary prewhitening procedure gives significant computational savings and, as we demonstrate next by the example of the BSS problem, can lead to a better performance if the sample size is small and the source pairs are possibly correlated.

5. Second-order blind identification with unitary prewhitening

As an interesting application of the proposed unitary prewhitening scheme, we consider the BSS problem. In particular, we propose a modification of the second-order blind identification (SOBI) procedure by Belochrani et al. [1]. It is important to note that the two most computationally expensive steps in the SOBI algorithm are the prewhitening and the joint approximate diagonalization [2]. The precise estimate of the computational complexity of the joint approximate diagonalization is not known and depends on the specific set of matrices that should be jointly diagonalized. However, based on extensive simulations, we observed that it is comparable to the complexity of the prewhitening of the complex-valued signals. The computational complexity of the prewhitening is defined by the complexities of the EVD and inversion operations, where the complexity of each operation is $\mathcal{O}(K^3)$. Thus, the prewhitening step is indeed a bottleneck of the SOBI algorithm, and by reducing the computational complexity of this step by a factor of four in the case of separation of complex-valued signals, the complexity of the overall SOBI procedure can be reduced at least by a factor of eight.

Following the steps of the original SOBI procedure [1] and taking into account the previously proposed prewhitening algorithm, the SOBI with unitary prewhitening can be defined by the following implementation:

- (1) Estimate the sample data covariance matrix $\widehat{\mathbf{R}}_y(0)$ and calculate the real-valued covariance matrix $\widehat{\mathbf{C}}$ using (14).
- (2) Estimate the noise variance $\hat{\sigma}^2$ using (18) and calculate the whitening matrix and whitened signals using (17) and (19), respectively.
- (3) Compute the sample covariance matrices $\widehat{\mathbf{R}}_z(\tau)$ for a fixed set of time lags $\tau \in \{\tau_i \mid i = 1, \dots, I\}$, where I is the total number of time lags.
- (4) Find the estimate of the unitary matrix \mathbf{U} (see (4)) by applying the joint approximate diagonalization procedure [2] to the set $\{\widehat{\mathbf{R}}_z(\tau_i) \mid i = 1, \dots, I\}$.
- (5) Using (4), estimate the mixing matrix \mathbf{A} , which we denote as $\widehat{\mathbf{A}}$.

6. Simulations

In this section, we numerically compare the performance of the SOBI with the proposed unitary prewhitening to that of the SOBI with conventional prewhitening [1]. The performances are compared in terms of the root-mean-square error (RMSE):

$$\text{RMSE} = \sqrt{\frac{1}{LMK} \sum_{l=1}^L \|\widehat{\mathbf{A}}(l) - \mathbf{A}\|_{\text{F}}^2}, \quad (20)$$

where $L = 100$ is the number of independent simulation runs, $\widehat{\mathbf{A}}(l)$ is the estimate of \mathbf{A} obtained from the l th run, and $\|\cdot\|_{\text{F}}^2$ stands for the Frobenius norm of matrix. Permutation and scaling of columns (which are inherent in the BSS problem) are fixed by means of a least-squares ordering and normalization of the columns of $\widehat{\mathbf{A}}(l)$. In particular, we first form an $M \times M$ distance matrix whose (m, n) th element contains the Euclidean distance between the m th column of \mathbf{A} and the n th column of $\widehat{\mathbf{A}}(l)$. The smallest element of this distance matrix determines the first match, and the respective row and column of this matrix are deleted. The process is then repeated with the reduced-size distance matrix.

A 10-element ULA with half-wavelength sensor spacing receives two signals in the presence of stationary complex white Gaussian noise. The source signals are narrowband far-field unit variance complex circular Gaussian, and arrive from different directions $\theta_1 = 2^\circ$ and $\theta_2 = 15^\circ$. The source waveforms are generated by filtering a complex circular white Gaussian process by autoregressive (AR) models of first order with the coefficients $a_1 = 0.85 \exp\{j0.5\}$ and $a_2 = 0.85 \exp\{j0.55\}$, respectively. Six time lags are used to compute six covariance matrices $\mathbf{R}_z(\tau_i)$, $i = 1, \dots, 6$ for each aforementioned method, and the single-sensor signal-to-noise ratio (SNR) is taken to be equal to 10 dB.

In our first example, the sources are assumed to be uncorrelated. However, the correlation between sources can be caused in this case by the finite sample size effect. Figure 1 shows the RMSEs of the aforementioned methods versus the number of snapshots.

In the second example, mutually correlated sources are assumed with the correlation coefficient equal to 0.99. Note that the assumption of spatially uncorrelated sources which enables us to use the conventional SOBI is violated in this example. Figure 2 shows the RMSEs of the tested methods versus the number of snapshots.

From the figures we observe that the SOBI with unitary prewhitening outperforms the SOBI with conventional prewhitening. This is especially true if the sample size is small or the pairs of source signals are correlated. The latter improvements take place due to the fact that FB averaging of the sample covariance matrix is used in the proposed scheme. However, the performance improvement may not be achieved if the number of correlated signals is larger than two. At the same time, the computational complexity of the prewhitening step for SOBI with

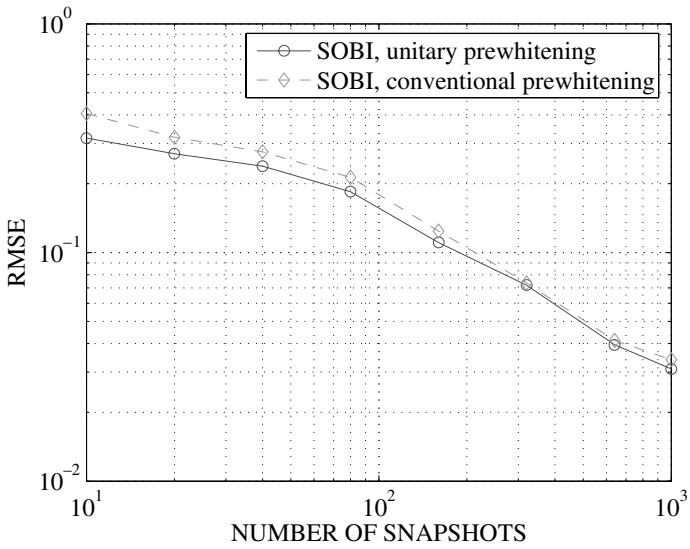


Figure 1. Spatial signature estimation RMSEs versus number of snapshots. Uncorrelated sources.

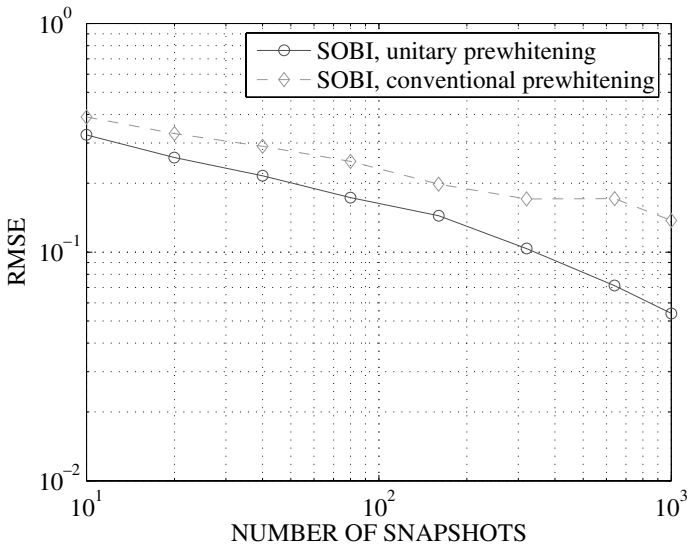


Figure 2. Spatial signature estimation RMSEs versus number of snapshots. Correlated pair of sources.

unitary prewhitening is reduced by approximately a factor of four, and the overall complexity reduced by a factor of eight.

7. Conclusions

A new prewhitening scheme has been proposed whereby a complex-valued EVD is equivalently substituted by the real-valued EVD. The computational complexity of the new prewhitening scheme compared to the conventional one can be reduced approximately by a factor of four. The proposed scheme is applied to the BSS problem, where the prewhitening step is a bottleneck step from the point of view of computational complexity. For this application, the improvement in the performance due to exploiting the proposed unitary prewhitening scheme is clearly shown by means of simulations. Such an improvement is especially pronounced in the cases of short sample size and correlated pairs of source signals.

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