Robust Adaptive Beamforming: Evolution of Approaches, Analysis and Comparison

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Outline

Introduction

- Adaptive Beamforming finds applications in many areas such as radar, sonar, wireless communications, etc.
- Conventional beamforming techniques assume
  - *the steering vector of the desired signal is known precisely*
  - *large number of snapshots (training sample size)*
  - *stationary training data set*
- In many practical situations there is mismatch between the presumed steering vector and the actual one!

Signal Model

The output of a narrowband beamformer

$$y(k) = \mathbf{w}^H \mathbf{x}(k)$$

where

$$\mathbf{x}(k) = \underbrace{s(k)\mathbf{p}}_{\text{signal}} + \underbrace{\mathbf{i}(k)}_{\text{interference}} + \underbrace{\mathbf{n}(k)}_{\text{noise}}$$

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Actual steering vector

$$\underbrace{\mathbf{a}}_{\text{actual}} = \underbrace{\mathbf{p}}_{\text{presumed}} + \underbrace{\mathbf{e}}_{\text{unknown mismatch}}$$

Maximum SINR criterion

$$\max_{\boldsymbol{w}} \mathrm{SINR}\,, \qquad \mathrm{SINR} = \frac{\sigma_{\mathrm{s}}^2 \left|\boldsymbol{w}^H(\boldsymbol{p}+\boldsymbol{e})\right|^2}{\boldsymbol{w}^H \boldsymbol{R}_{\mathrm{i+n}} \boldsymbol{w}}$$

Interference-plus-noise covariance matrix

$$\boldsymbol{R}_{\mathrm{i+n}} = \mathrm{E}\left\{(\boldsymbol{i}(k)+\boldsymbol{n}(k))\,(\boldsymbol{i}(k)+\boldsymbol{n}(k))^H\right\}$$

Note

- In practice, $\boldsymbol{R}_{\mathrm{i+n}}$ is unavailable
- Sample estimate $\hat{\boldsymbol{R}} \triangleq \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{x}(k)\boldsymbol{x}^H(k)$ is used

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Worst-Case Performance Optimization

The essence of this approach is to

- Maintain a distortionless response towards a continuum of steering vectors that belong to a certain uncertainty set
- Guarantee that the distortionless response is maintained in the worst case
- Model the uncertainty about the mismatch vector using
  - spherical uncertainty set [Vorobyov, Gershman, Luo '03]
  - elliptical uncertainty set [Lorenz and Boyd '05]

Problem Formulation and Main Result

- The spherical uncertainty set is (for some known $\varepsilon > 0$)

$$\|\boldsymbol{e}\| \leq \varepsilon$$

The robust MVDR beamformeing problem is formulated as

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \quad \text{s. t.} \quad |\boldsymbol{w}^H(\boldsymbol{p}+\boldsymbol{e})| \geq 1, \quad \forall \quad \|\boldsymbol{e}\| \leq \varepsilon$$

- *Result 1* [Vorobyov, Gershman, Luo '03]: Infinite number of non-convex constraints

$$|\boldsymbol{w}^H(\boldsymbol{p}+\boldsymbol{e})| \geq 1, \quad \forall \quad \|\boldsymbol{e}\| \leq \varepsilon$$

is equvalent to a single convex constraint

$$\varepsilon\|\boldsymbol{w}\| \leq \boldsymbol{w}^H\boldsymbol{p} - 1$$

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Robust MVDR Beamforming

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This is so-called convex second order cone (SOC) programming problem! It can be easily solved!

Problem Formulation and Main Results

- The probabilistically-constrained beamformer guarantees that the distortionless response is maintained *with a certain "sufficient" probability*

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- *Result 2* [Vorobyov, Chen, Gershman '08]: For Gaussian mismatch

$$\boldsymbol{e} \sim \mathcal{N}_{\mathcal{C}}(\boldsymbol{0}, \boldsymbol{C}_{\boldsymbol{e}})$$

the probabilistic constraint is tightly approximated by the deterministic constraint

$$\sqrt{-\ln(1-p_0)}\ \|\boldsymbol{C}_{\boldsymbol{e}}^{1/2}\mathbf{w}\| \leq \mathbf{w}^H\mathbf{p} - 1$$

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Probabilistically-Constrained Beamformer

- *Result 3* [Vorobyov, Chen, Gershman '08]: For mismatch with the worst-case distribution the probabilistic constraint is tightly approximated by the deterministic constraint

$$\frac{1}{\sqrt{1-p_0}}\,\|\boldsymbol{C}_{\boldsymbol{e}}^{1/2}\mathbf{w}\| \leq \mathbf{w}^H\mathbf{p} - 1$$

Moreover, the worst-case distribution is discrete.

- *The problem is equivalent to that of the worst-case based robust adaptive beamforming* if $\boldsymbol{C}_{\boldsymbol{e}} = (\sigma_{\boldsymbol{e}}^2/M)\boldsymbol{I}$.

For the worst-case mismatch distribution: $\varepsilon = \sigma_{\boldsymbol{e}}\sqrt{\frac{1}{M(1-p_0)}}$

For Gaussian mismatch: $\varepsilon = \sigma_{\boldsymbol{e}}\sqrt{\frac{-\ln(1-p_0)}{M}}$

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Analysis and a New Idea

Problems with previous approaches

- If mismatch is Gaussian, its norm is Chi-square distributed (not norm bounded)
- Over/under estimation of the parameters, e.g. ε, may lead to degradation in performance

Essence of a new approache

Estimate the mismatch vector and form the beam using the corrected steering vector [Hassanien, Vorobyov, Wong '08]

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Problem Formulation

- First maximize the beamformer output SINR by solving the optimization problem

$$\min_{\mathbf{w}} \ \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad \text{subject to} \quad \mathbf{w}^H(\mathbf{p} + \mathbf{e}) = 1$$

- Solution

$$\mathbf{w}(\mathbf{e}) = \frac{\hat{\mathbf{R}}^{-1}(\mathbf{p} + \mathbf{e})}{(\mathbf{p} + \mathbf{e})^H \hat{\mathbf{R}}^{-1}(\mathbf{p} + \mathbf{e})}$$

- *The beamformer output power*

$$P(\mathbf{e}) = \frac{1}{(\mathbf{p} + \mathbf{e})^H \hat{\mathbf{R}}^{-1}(\mathbf{p} + \mathbf{e})}$$

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Problem Formulation and Difficulties

Estimate the unknown mismatch vector **e** by maximizing the beamformer output power

$$\min_{\mathbf{e}} (\mathbf{p}+\mathbf{e})^H \hat{\mathbf{R}}^{-1} (\mathbf{p}+\mathbf{e}) \quad \text{s. t.} \quad \|\mathbf{p}+\mathbf{e}\| = \sqrt{M}$$

Two difficulties

- The corrected vector $\mathbf{p}+\hat{\mathbf{e}}$ might converge to a vector associated with interference
- Non-convex constraint!

Problem Formulation and Difficulties

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Robust Adaptive Beamforming Using SQP

- To avoid first difficulty, enforce $\mathbf{p}+\mathbf{e}$ to belong to a subspace that is spanned by the actual steering vector

$$\mathbf{P}_{\mathbf{p}}^{\perp}(\mathbf{p}+\mathbf{e}) = \mathbf{0}$$

  $\mathbf{P}_{\mathbf{p}}^{\perp} \triangleq \mathbf{I} - \mathbf{U}\mathbf{U}^H$ is a projection onto a subspace that is orthogonal to the actual steering vector

- $\mathbf{U} \triangleq [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_K]$, $\{\mathbf{u}_k\}_{k=1}^{K}$ are K principal eigenvectors of

$$\mathbf{C} \triangleq \int_{\Theta} \mathbf{p}(\theta)\mathbf{p}^H(\theta)\, d\theta$$

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Robust Adaptive Beamforming Using SQP (Cont'd)

- *Result 4* [Hassanien, Vorobyov, Wong '08]: The initial optimization problem is equivalent to the problem

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$$\text{subject to} \quad \mathbf{P}_{\mathbf{p}}^{\perp}(\mathbf{p}+\mathbf{e}) = \mathbf{0}$$
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- How to get rid of non-convexity?

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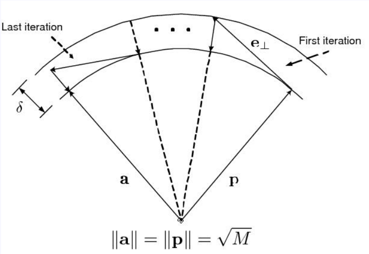
Iterative Solution



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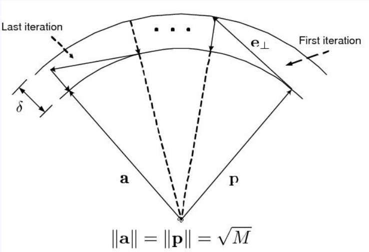


$$\begin{aligned} \min_{\mathbf{e}_\perp} \quad & (\mathbf{p}+\mathbf{e}_\perp)^H \hat{\mathbf{R}}^{-1} (\mathbf{p}+\mathbf{e}_\perp) \\ \text{subject to} \quad & \mathbf{P}_{\mathbf{p}}^\perp (\mathbf{p}+\mathbf{e}) = \mathbf{0} \\ & \|\mathbf{p}+\mathbf{e}_\perp\| \leq \sqrt{M} + \delta \\ & \mathbf{p}^H \mathbf{e}_\perp = 0 \end{aligned}$$

Iterative Algorithm

Algorithm:

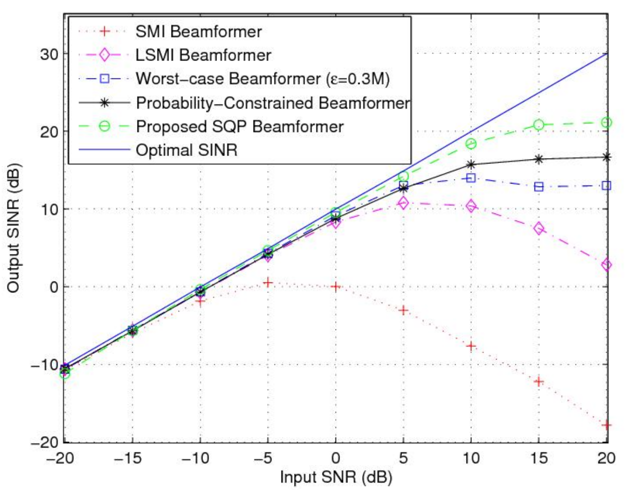
1. Estimate $\mathbf{e}_\perp$ by solving the problem in previous slide
2. If $\|\mathbf{e}_\perp\| =$ "small", go to Step 5.
3. Update the presumed steering vector $\mathbf{p} = \mathbf{p} + \mathbf{e}_\perp$.
4. Project the updated steering vector back to the sphere $\mathbf{p} = \left(\sqrt{M}/\|\mathbf{p}\|\right)\mathbf{p}$, then go to Step 1.
5. Calculate the robust adaptive beamformer weights

$$\mathbf{w}_{\mathrm{SQP}} = \frac{\hat{\mathbf{R}}^{-1}\mathbf{p}}{\mathbf{p}^H\hat{\mathbf{R}}^{-1}\mathbf{p}},$$

Simulation Setup

- $M = 10$ sensors spaced half wavelength apart. $N = 100$ data snapshots.
- Desired signal is assumed to impinge on the array from direction $\theta_p = 5^\circ$
- Two interfering sources with DOAs -50° and -20°; INR $= 30$ dB.
- Look direction mismatch:
  actual DOA is uniformly drawn from $[1^\circ\ 9^\circ]$
- Array perturbation:
  Sensors are assumed to be displaced from its original location and the displacement is drawn uniformly from the set $[-0.05,\ 0.05]$ measured in wavelength.

Simulation Results



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