1. Show that

$$f(n) = \frac{3^{2^{n-1}}+1}{3^{2^{n-1}}-1}$$

satisfies, for every $n \geq 1$,

$$f(n) = \frac{1}{2}(f(n-1) + \frac{1}{f(n-1)})$$

with $f(1) = 2$.

Proof:

1)

$$f(1) = 2 = \frac{3+1}{3-1}$$

.

2) Assume that $f(k)$ satisfies the recurrence, i.e.

$$f(k) = \frac{1}{2}(f(k-1) + \frac{1}{f(k-1)})$$

.

3)

$$\begin{aligned}
f(k+1) &= \frac{1}{2}(\frac{3^{2^{k-1}}+1}{3^{2^{k-1}}-1} + \frac{3^{2^{k-1}}-1}{3^{2^{k-1}}+1}) && (1)\\
&= \frac{1}{2}(\frac{(3^{2^{k-1}}+1)^2}{(3^{2^{k-1}})^2-1} + \frac{(3^{2^{k-1}}-1)^2}{(3^{2^{k-1}})^2-1}) && (2)\\
&= \frac{1}{2}(\frac{2 \times 3^{2^k}+2}{3^{2^k}-1}) && (3)\\
&= \frac{3^{2^k}+1}{3^{2^k}-1} && (4)
\end{aligned}$$

.

The claim follows from 1), 2) and 3), *quod erat demonstrandum.*