Lecture 7: Bayesian Smoother, Gaussian and Particle Smoothers

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March 16, 2016

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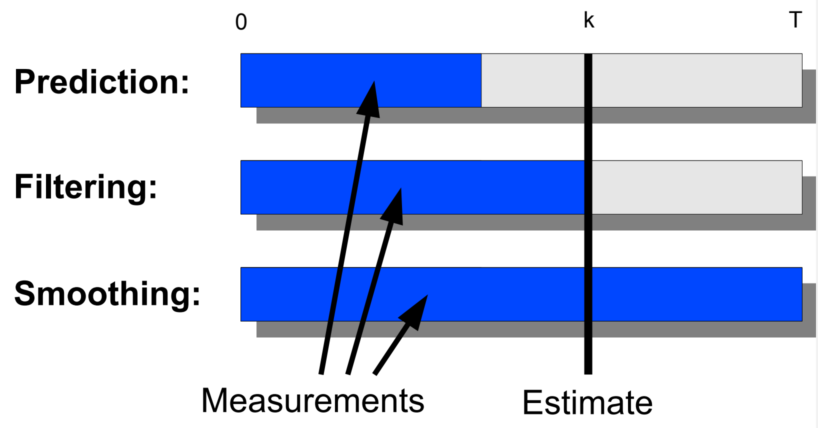
Filtering, Prediction and Smoothing



Types of Smoothing Problems

- Fixed-interval smoothing: estimate states on interval $[0, T]$ given measurements on the same interval.
- Fixed-point smoothing: estimate state at a fixed point of time in the past.
- Fixed-lag smoothing: estimate state at a fixed delay in the past.
- Here we shall only consider fixed-interval smoothing, the others can be quite easily derived from it.

Examples of Smoothing Problems

- Given all the radar measurements of a rocket (or missile) trajectory, what was the exact place of launch?
- Estimate the whole trajectory of a car based on GPS measurements to calibrate the inertial navigation system accurately.
- What was the history of chemical/combustion/other process given a batch of measurements from it?
- Remove noise from audio signal by using smoother to estimate the true audio signal under the noise.
- Smoothing solution also arises in EM algorithm for estimating the parameters of a state space model.

Bayesian Smoothing Algorithms

- Linear Gaussian models
  - Rauch-Tung-Striebel smoother (RTSS).
  - Two-filter smoother.
- Non-linear Gaussian models
  - Extended Rauch-Tung-Striebel smoother (ERTSS).
  - Unscented Rauch-Tung-Striebel smoother (URTSS).
  - Statistically linearized Rauch-Tung-Striebel smoother (SLRTSS).
  - Gaussian Rauch-Tung-Striebel smoothers (GRTSS), cubature, Gauss-Hermite, Bayes-Hermite, Monte Carlo.
  - Two-filter versions of the above.
- Non-linear non-Gaussian models
  - Particle smoothers.
  - Rao-Blackwellized particle smoothers.
  - Grid based smoothers.

Problem Formulation

- Probabilistic state space model:

$$\text{measurement model: } \mathbf{y}_k \sim p(\mathbf{y}_k \mid \mathbf{x}_k)$$
$$\text{dynamic model: } \mathbf{x}_k \sim p(\mathbf{x}_k \mid \mathbf{x}_{k-1})$$

- Assume that the filtering distributions $p(\mathbf{x}_k \mid \mathbf{y}_{1:k})$ have already been computed for all $k = 0, \ldots, T$.
- We want recursive equations of computing the smoothing distribution for all $k < T$:

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:T}).$$

- The recursion will go backwards in time, because on the last step, the filtering and smoothing distributions coincide:

$$p(\mathbf{x}_T \mid \mathbf{y}_{1:T}).$$

Derivation of Formal Smoothing Equations [1/2]

- The key: due to the Markov properties of state we have:

$$p(\mathbf{x}_k \,|\, \mathbf{x}_{k+1}, \mathbf{y}_{1:T}) = p(\mathbf{x}_k \,|\, \mathbf{x}_{k+1}, \mathbf{y}_{1:k})$$

- Thus we get:

$$\begin{aligned}
p(\mathbf{x}_k \,|\, \mathbf{x}_{k+1}, \mathbf{y}_{1:T}) &= p(\mathbf{x}_k \,|\, \mathbf{x}_{k+1}, \mathbf{y}_{1:k}) \\
&= \frac{p(\mathbf{x}_k, \mathbf{x}_{k+1} \,|\, \mathbf{y}_{1:k})}{p(\mathbf{x}_{k+1} \,|\, \mathbf{y}_{1:k})} \\
&= \frac{p(\mathbf{x}_{k+1} \,|\, \mathbf{x}_k, \mathbf{y}_{1:k}) \, p(\mathbf{x}_k \,|\, \mathbf{y}_{1:k})}{p(\mathbf{x}_{k+1} \,|\, \mathbf{y}_{1:k})} \\
&= \frac{p(\mathbf{x}_{k+1} \,|\, \mathbf{x}_k) \, p(\mathbf{x}_k \,|\, \mathbf{y}_{1:k})}{p(\mathbf{x}_{k+1} \,|\, \mathbf{y}_{1:k})}.
\end{aligned}$$

Derivation of Formal Smoothing Equations [2/2]

- Assuming that the smoothing distribution of the next step $p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T})$ is available, we get

$$\begin{aligned}
p(\mathbf{x}_k, \mathbf{x}_{k+1} \mid \mathbf{y}_{1:T}) &= p(\mathbf{x}_k \mid \mathbf{x}_{k+1}, \mathbf{y}_{1:T}) \, p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T}) \\
&= p(\mathbf{x}_k \mid \mathbf{x}_{k+1}, \mathbf{y}_{1:k}) \, p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T}) \\
&= \frac{p(\mathbf{x}_{k+1} \mid \mathbf{x}_k) \, p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) \, p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T})}{p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:k})}
\end{aligned}$$

- Integrating over $\mathbf{x}_{k+1}$ gives

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:T}) = p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) \int \left[\frac{p(\mathbf{x}_{k+1} \mid \mathbf{x}_k) \, p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T})}{p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:k})} \right] \mathrm{d}\mathbf{x}_{k+1}$$

Bayesian Smoothing Equations

The Bayesian smoothing equations consist of prediction step and backward update step:

$$p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:k}) = \int p(\mathbf{x}_{k+1} \mid \mathbf{x}_k) \, p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) \, \mathrm{d}\mathbf{x}_k$$

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:T}) = p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) \int \left[\frac{p(\mathbf{x}_{k+1} \mid \mathbf{x}_k) \, p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T})}{p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:k})} \right] \mathrm{d}\mathbf{x}_{k+1}$$

The recursion is started from the filtering (and smoothing) distribution of the last time step $p(\mathbf{x}_T \mid \mathbf{y}_{1:T})$.

Linear-Gaussian Smoothing Problem

- Gaussian driven linear model, i.e., Gauss-Markov model:

$$\mathbf{x}_k = \mathbf{A}_{k-1}\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1}$$
$$\mathbf{y}_k = \mathbf{H}_k\,\mathbf{x}_k + \mathbf{r}_k,$$

- In probabilistic terms the model is

$$p(\mathbf{x}_k \,|\, \mathbf{x}_{k-1}) = \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{A}_{k-1}\,\mathbf{x}_{k-1}, \mathbf{Q}_{k-1})$$
$$p(\mathbf{y}_k \,|\, \mathbf{x}_k) = \mathrm{N}(\mathbf{y}_k \,|\, \mathbf{H}_k\,\mathbf{x}_k, \mathbf{R}_k).$$

- Kalman filter can be used for computing all the Gaussian filtering distributions:

$$p(\mathbf{x}_k \,|\, \mathbf{y}_{1:k}) = \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k, \mathbf{P}_k).$$

- Gaussian probability density

$$\mathrm{N}(\mathbf{x}\,|\,\mathbf{m},\mathbf{P}) = \frac{1}{(2\,\pi)^{n/2}\,|\mathbf{P}|^{1/2}}\exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\,\mathbf{P}^{-1}\,(\mathbf{x}-\mathbf{m})\right),$$

- Let $\mathbf{x}$ and $\mathbf{y}$ have the Gaussian densities

$$p(\mathbf{x}) = \mathrm{N}(\mathbf{x}\,|\,\mathbf{m},\mathbf{P}), \qquad p(\mathbf{y}\,|\,\mathbf{x}) = \mathrm{N}(\mathbf{y}\,|\,\mathbf{H}\,\mathbf{x},\mathbf{R}),$$

- Then the joint and marginal distributions are

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix}\mathbf{m}\\ \mathbf{H}\,\mathbf{m}\end{pmatrix},\begin{pmatrix}\mathbf{P} & \mathbf{P}\,\mathbf{H}^T\\ \mathbf{H}\,\mathbf{P} & \mathbf{H}\,\mathbf{P}\,\mathbf{H}^T+\mathbf{R}\end{pmatrix}\right)$$

$$\mathbf{y} \sim \mathrm{N}(\mathbf{H}\,\mathbf{m},\mathbf{H}\,\mathbf{P}\,\mathbf{H}^T+\mathbf{R}).$$

RTS: Derivation Preliminaries (cont.)

- If the random variables $\mathbf{x}$ and $\mathbf{y}$ have the joint Gaussian probability density

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} \right),$$

- Then the marginal and conditional densities of $\mathbf{x}$ and $\mathbf{y}$ are given as follows:

$$\begin{aligned} \mathbf{x} &\sim \mathrm{N}(\mathbf{a}, \mathbf{A}) \\ \mathbf{y} &\sim \mathrm{N}(\mathbf{b}, \mathbf{B}) \\ \mathbf{x} \mid \mathbf{y} &\sim \mathrm{N}(\mathbf{a} + \mathbf{C}\,\mathbf{B}^{-1}\,(\mathbf{y} - \mathbf{b}), \mathbf{A} - \mathbf{C}\,\mathbf{B}^{-1}\mathbf{C}^T) \\ \mathbf{y} \mid \mathbf{x} &\sim \mathrm{N}(\mathbf{b} + \mathbf{C}^T\,\mathbf{A}^{-1}\,(\mathbf{x} - \mathbf{a}), \mathbf{B} - \mathbf{C}^T\,\mathbf{A}^{-1}\,\mathbf{C}). \end{aligned}$$

- By the Gaussian distribution computation rules we get

$$\begin{aligned}
p(\mathbf{x}_k, \mathbf{x}_{k+1} \mid \mathbf{y}_{1:k}) &= p(\mathbf{x}_{k+1} \mid \mathbf{x}_k)\, p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) \\
&= \mathrm{N}(\mathbf{x}_{k+1} \mid \mathbf{A}_k\, \mathbf{x}_k, \mathbf{Q}_k)\, \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k, \mathbf{P}_k) \\
&= \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k+1} \end{bmatrix} \,\middle|\, \mathbf{m}_1, \mathbf{P}_1 \right),
\end{aligned}$$

where

$$\mathbf{m}_1 = \begin{pmatrix} \mathbf{m}_k \\ \mathbf{A}_k\, \mathbf{m}_k \end{pmatrix}, \qquad \mathbf{P}_1 = \begin{pmatrix} \mathbf{P}_k & \mathbf{P}_k\, \mathbf{A}_k^T \\ \mathbf{A}_k\, \mathbf{P}_k & \mathbf{A}_k\, \mathbf{P}_k\, \mathbf{A}_k^T + \mathbf{Q}_k \end{pmatrix}.$$

Derivation of Rauch-Tung-Striebel Smoother [2/4]

- By conditioning rule of Gaussian distribution we get

$$\begin{aligned} p(\mathbf{x}_k \mid \mathbf{x}_{k+1}, \mathbf{y}_{1:T}) &= p(\mathbf{x}_k \mid \mathbf{x}_{k+1}, \mathbf{y}_{1:k}) \\ &= \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_2, \mathbf{P}_2), \end{aligned}$$

where

$$\begin{aligned} \mathbf{G}_k &= \mathbf{P}_k \, \mathbf{A}_k^T \, (\mathbf{A}_k \, \mathbf{P}_k \, \mathbf{A}_k^T + \mathbf{Q}_k)^{-1} \\ \mathbf{m}_2 &= \mathbf{m}_k + \mathbf{G}_k \, (\mathbf{x}_{k+1} - \mathbf{A}_k \, \mathbf{m}_k) \\ \mathbf{P}_2 &= \mathbf{P}_k - \mathbf{G}_k \, (\mathbf{A}_k \, \mathbf{P}_k \, \mathbf{A}_k^T + \mathbf{Q}_k) \, \mathbf{G}_k^T. \end{aligned}$$

Derivation of Rauch-Tung-Striebel Smoother [3/4]

- The joint distribution of $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ given all the data is

$$
\begin{aligned}
p(\mathbf{x}_{k+1}, \mathbf{x}_k \mid \mathbf{y}_{1:T}) &= p(\mathbf{x}_k \mid \mathbf{x}_{k+1}, \mathbf{y}_{1:T})\, p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T}) \\
&= \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_2, \mathbf{P}_2)\; \mathrm{N}(\mathbf{x}_{k+1} \mid \mathbf{m}^s_{k+1}, \mathbf{P}^s_{k+1}) \\
&= \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{x}_k \end{bmatrix} \,\middle|\, \mathbf{m}_3, \mathbf{P}_3 \right)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{m}_3 &= \begin{pmatrix} \mathbf{m}^s_{k+1} \\ \mathbf{m}_k + \mathbf{G}_k\,(\mathbf{m}^s_{k+1} - \mathbf{A}_k\,\mathbf{m}_k) \end{pmatrix} \\
\mathbf{P}_3 &= \begin{pmatrix} \mathbf{P}^s_{k+1} & \mathbf{P}^s_{k+1}\,\mathbf{G}_k^T \\ \mathbf{G}_k\,\mathbf{P}^s_{k+1} & \mathbf{G}_k\,\mathbf{P}^s_{k+1}\,\mathbf{G}_k^T + \mathbf{P}_2 \end{pmatrix}.
\end{aligned}
$$

- The marginal mean and covariance are thus given as

$$\begin{aligned}\mathbf{m}_k^s &= \mathbf{m}_k + \mathbf{G}_k\,(\mathbf{m}_{k+1}^s - \mathbf{A}_k\,\mathbf{m}_k)\\ \mathbf{P}_k^s &= \mathbf{P}_k + \mathbf{G}_k\,(\mathbf{P}_{k+1}^s - \mathbf{A}_k\,\mathbf{P}_k\,\mathbf{A}_k^T - \mathbf{Q}_k)\,\mathbf{G}_k^T.\end{aligned}$$

- The smoothing distribution is then Gaussian with the above mean and covariance:

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:T}) = \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k^s, \mathbf{P}_k^s),$$

Rauch-Tung-Striebel Smoother

Backward recursion equations for the smoothed means $\mathbf{m}_k^s$ and covariances $\mathbf{P}_k^s$:

$$\begin{aligned}
\mathbf{m}_{k+1}^- &= \mathbf{A}_k \, \mathbf{m}_k \\
\mathbf{P}_{k+1}^- &= \mathbf{A}_k \, \mathbf{P}_k \, \mathbf{A}_k^T + \mathbf{Q}_k \\
\mathbf{G}_k &= \mathbf{P}_k \, \mathbf{A}_k^T \, [\mathbf{P}_{k+1}^-]^{-1} \\
\mathbf{m}_k^s &= \mathbf{m}_k + \mathbf{G}_k \, [\mathbf{m}_{k+1}^s - \mathbf{m}_{k+1}^-] \\
\mathbf{P}_k^s &= \mathbf{P}_k + \mathbf{G}_k \, [\mathbf{P}_{k+1}^s - \mathbf{P}_{k+1}^-] \, \mathbf{G}_k^T,
\end{aligned}$$

- $\mathbf{m}_k$ and $\mathbf{P}_k$ are the mean and covariance computed by the Kalman filter.
- The recursion is started from the last time step T, with $\mathbf{m}_T^s = \mathbf{m}_T$ and $\mathbf{P}_T^s = \mathbf{P}_T$.

RTS Smoother: Car Tracking Example

The dynamic model of the car tracking model from the first & third lectures was:

$$\begin{pmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x_{k-1} \\ y_{k-1} \\ \dot{x}_{k-1} \\ \dot{y}_{k-1} \end{pmatrix} + \mathbf{q}_{k-1}$$

where $\mathbf{q}_k$ is zero mean with a covariance matrix $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} q_1^c \, \Delta t^3/3 & 0 & q_1^c \, \Delta t^2/2 & 0 \\ 0 & q_2^c \, \Delta t^3/3 & 0 & q_2^c \, \Delta t^2/2 \\ q_1^c \, \Delta t^2/2 & 0 & q_1^c \, \Delta t & 0 \\ 0 & q_2^c \, \Delta t^2/2 & 0 & q_2^c \, \Delta t \end{pmatrix}$$

Non-Linear Smoothing Problem

- Non-linear Gaussian state space model:

$$\begin{aligned}\mathbf{x}_k &= \mathbf{f}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1} \\ \mathbf{y}_k &= \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k,\end{aligned}$$

- We want to compute Gaussian approximations to the smoothing distributions:

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:T}) \approx \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k^s, \mathbf{P}_k^s).$$

Extended Rauch-Tung-Striebel Smoother Derivation

- The approximate joint distribution of $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ is

$$p(\mathbf{x}_k, \mathbf{x}_{k+1} \,|\, \mathbf{y}_{1:k}) = \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k+1} \end{bmatrix} \,\Big|\, \mathbf{m}_1, \mathbf{P}_1 \right),$$

where

$$\begin{aligned}
\mathbf{m}_1 &= \begin{pmatrix} \mathbf{m}_k \\ \mathbf{f}(\mathbf{m}_k) \end{pmatrix} \\
\mathbf{P}_1 &= \begin{pmatrix} \mathbf{P}_k & \mathbf{P}_k \, \mathbf{F}_x^T(\mathbf{m}_k) \\ \mathbf{F}_x(\mathbf{m}_k) \, \mathbf{P}_k & \mathbf{F}_x(\mathbf{m}_k) \, \mathbf{P}_k \, \mathbf{F}_x^T(\mathbf{m}_k) + \mathbf{Q}_k \end{pmatrix}.
\end{aligned}$$

- The rest of the derivation is analogous to the linear RTS smoother.

Extended Rauch-Tung-Striebel Smoother

The equations for the extended RTS smoother are

$$
\begin{aligned}
\mathbf{m}_{k+1}^{-} &= \mathbf{f}(\mathbf{m}_k) \\
\mathbf{P}_{k+1}^{-} &= \mathbf{F}_{\mathbf{x}}(\mathbf{m}_k)\, \mathbf{P}_k\, \mathbf{F}_{\mathbf{x}}^{T}(\mathbf{m}_k) + \mathbf{Q}_k \\
\mathbf{G}_k &= \mathbf{P}_k\, \mathbf{F}_{\mathbf{x}}^{T}(\mathbf{m}_k)\, [\mathbf{P}_{k+1}^{-}]^{-1} \\
\mathbf{m}_k^{s} &= \mathbf{m}_k + \mathbf{G}_k\, [\mathbf{m}_{k+1}^{s} - \mathbf{m}_{k+1}^{-}] \\
\mathbf{P}_k^{s} &= \mathbf{P}_k + \mathbf{G}_k\, [\mathbf{P}_{k+1}^{s} - \mathbf{P}_{k+1}^{-}]\, \mathbf{G}_k^{T},
\end{aligned}
$$

where the matrix $\mathbf{F}_{\mathbf{x}}(\mathbf{m}_k)$ is the Jacobian matrix of $\mathbf{f}(\mathbf{x})$ evaluated at $\mathbf{m}_k$.

Statistically Linearized Rauch-Tung-Striebel Smoother Derivation

- With statistical linearization we get the approximation

$$p(\mathbf{x}_k, \mathbf{x}_{k+1} \mid \mathbf{y}_{1:k}) = \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k+1} \end{bmatrix} \,\middle|\, \mathbf{m}_1, \mathbf{P}_1 \right),$$

where

$$\mathbf{m}_1 = \begin{pmatrix} \mathbf{m}_k \\ \mathrm{E}[\mathbf{f}(\mathbf{x}_k)] \end{pmatrix}$$

$$\mathbf{P}_1 = \begin{pmatrix} \mathbf{P}_k & \mathrm{E}[\mathbf{f}(\mathbf{x}_k)\,\delta\mathbf{x}_k^T]^T \\ \mathrm{E}[\mathbf{f}(\mathbf{x}_k)\,\delta\mathbf{x}_k^T] & \mathrm{E}[\mathbf{f}(\mathbf{x}_k)\,\delta\mathbf{x}_k^T]\,\mathbf{P}_k^{-1}\,\mathrm{E}[\mathbf{f}(\mathbf{x}_k)\,\delta\mathbf{x}_k^T]^T + \mathbf{Q}_k \end{pmatrix}.$$

- The expectations are taken with respect to filtering distribution of $\mathbf{x}_k$.
- The derivation proceeds as with linear RTS smoother.

Statistically Linearized Rauch-Tung-Striebel Smoother

The equations for the statistically linearized RTS smoother are

$$
\begin{aligned}
\mathbf{m}_{k+1}^{-} &= \mathrm{E}[\mathbf{f}(\mathbf{x}_k)] \\
\mathbf{P}_{k+1}^{-} &= \mathrm{E}[\mathbf{f}(\mathbf{x}_k)\,\delta\mathbf{x}_k^T]\,\mathbf{P}_k^{-1}\;\mathrm{E}[\mathbf{f}(\mathbf{x}_k)\,\delta\mathbf{x}_k^T]^T + \mathbf{Q}_k \\
\mathbf{G}_k &= \mathrm{E}[\mathbf{f}(\mathbf{x}_k)\,\delta\mathbf{x}_k^T]^T[\mathbf{P}_{k+1}^{-}]^{-1} \\
\mathbf{m}_k^s &= \mathbf{m}_k + \mathbf{G}_k\,[\mathbf{m}_{k+1}^s - \mathbf{m}_{k+1}^{-}] \\
\mathbf{P}_k^s &= \mathbf{P}_k + \mathbf{G}_k\,[\mathbf{P}_{k+1}^s - \mathbf{P}_{k+1}^{-}]\,\mathbf{G}_k^T,
\end{aligned}
$$

where the expectations are taken with respect to the filtering distribution $\mathbf{x}_k \sim \mathrm{N}(\mathbf{m}_k, \mathbf{P}_k)$.

Gaussian Rauch-Tung-Striebel Smoother Derivation

- With Gaussian moment matching we get the approximation

$$p(\mathbf{x}_k, \mathbf{x}_{k+1} \mid \mathbf{y}_{1:k}) = \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k+1} \end{bmatrix} \,\middle|\, \begin{bmatrix} \mathbf{m}_k \\ \mathbf{m}^-_{k+1} \end{bmatrix}, \begin{bmatrix} \mathbf{P}_k & \mathbf{D}_{k+1} \\ \mathbf{D}^T_{k+1} & \mathbf{P}^-_{k+1} \end{bmatrix} \right),$$

where

$$\begin{aligned}
\mathbf{m}^-_{k+1} &= \int \mathbf{f}(\mathbf{x}_k)\, \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k, \mathbf{P}_k)\, d\mathbf{x}_k \\
\mathbf{P}^-_{k+1} &= \int [\mathbf{f}(\mathbf{x}_k) - \mathbf{m}^-_{k+1}]\, [\mathbf{f}(\mathbf{x}_k) - \mathbf{m}^-_{k+1}]^T \\
&\qquad \times \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k, \mathbf{P}_k)\, d\mathbf{x}_k + \mathbf{Q}_k \\
\mathbf{D}_{k+1} &= \int [\mathbf{x}_k - \mathbf{m}_k]\, [\mathbf{f}(\mathbf{x}_k) - \mathbf{m}^-_{k+1}]^T \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k, \mathbf{P}_k)\, d\mathbf{x}_k.
\end{aligned}$$

Gaussian Rauch-Tung-Striebel Smoother

The equations for the Gaussian RTS smoother are

$$
\begin{aligned}
\mathbf{m}_{k+1}^{-} &= \int \mathbf{f}(\mathbf{x}_k)\,\mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k, \mathbf{P}_k)\,d\mathbf{x}_k \\
\mathbf{P}_{k+1}^{-} &= \int [\mathbf{f}(\mathbf{x}_k) - \mathbf{m}_{k+1}^{-}]\,[\mathbf{f}(\mathbf{x}_k) - \mathbf{m}_{k+1}^{-}]^T \\
&\qquad \times \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k, \mathbf{P}_k)\,d\mathbf{x}_k + \mathbf{Q}_k \\
\mathbf{D}_{k+1} &= \int [\mathbf{x}_k - \mathbf{m}_k]\,[\mathbf{f}(\mathbf{x}_k) - \mathbf{m}_{k+1}^{-}]^T \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k, \mathbf{P}_k)\,d\mathbf{x}_k \\
\mathbf{G}_k &= \mathbf{D}_{k+1}\,[\mathbf{P}_{k+1}^{-}]^{-1} \\
\mathbf{m}_k^s &= \mathbf{m}_k + \mathbf{G}_k\,(\mathbf{m}_{k+1}^s - \mathbf{m}_{k+1}^{-}) \\
\mathbf{P}_k^s &= \mathbf{P}_k + \mathbf{G}_k\,(\mathbf{P}_{k+1}^s - \mathbf{P}_{k+1}^{-})\,\mathbf{G}_k^T.
\end{aligned}
$$

Cubature Smoother Derivation [1/2]

- Recall the 3rd order spherical Gaussian integral rule:

$$\int \mathbf{g}(\mathbf{x}) \, \mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P}) \, d\mathbf{x} \approx \frac{1}{2n} \sum_{i=1}^{2n} \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi^{(i)}),$$

where

$$\xi^{(i)} = \begin{cases} \sqrt{n} \, \mathbf{e}_i & , \quad i = 1, \ldots, n \\ -\sqrt{n} \, \mathbf{e}_{i-n} & , \quad i = n+1, \ldots, 2n, \end{cases}$$

where $\mathbf{e}_i$ denotes a unit vector to the direction of coordinate axis i.

Cubature Smoother Derivation [2/2]

- We get the approximation

$$p(\mathbf{x}_k, \mathbf{x}_{k+1} \mid \mathbf{y}_{1:k}) = \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k+1} \end{bmatrix} \,\middle|\, \begin{bmatrix} \mathbf{m}_k \\ \mathbf{m}_{k+1}^- \end{bmatrix}, \begin{bmatrix} \mathbf{P}_k & \mathbf{D}_{k+1} \\ \mathbf{D}_{k+1}^T & \mathbf{P}_{k+1}^- \end{bmatrix} \right),$$

where

$$\begin{aligned}
\mathcal{X}_k^{(i)} &= \mathbf{m}_k + \sqrt{\mathbf{P}_k}\,\xi^{(i)} \\
\mathbf{m}_{k+1}^- &= \frac{1}{2n} \sum_{i=1}^{2n} \mathbf{f}(\mathcal{X}_k^{(i)}) \\
\mathbf{P}_{k+1}^- &= \frac{1}{2n} \sum_{i=1}^{2n} [\mathbf{f}(\mathcal{X}_k^{(i)}) - \mathbf{m}_{k+1}^-]\,[\mathbf{f}(\mathcal{X}_k^{(i)}) - \mathbf{m}_{k+1}^-]^T + \mathbf{Q}_k \\
\mathbf{D}_{k+1} &= \frac{1}{2n} \sum_{i=1}^{2n} [\mathcal{X}_k^{(i)} - \mathbf{m}_k]\,[\mathbf{f}(\mathcal{X}_k^{(i)}) - \mathbf{m}_{k+1}^-]^T.
\end{aligned}$$

Cubature Rauch-Tung-Striebel Smoother

1. Form the sigma points:

$$\mathcal{X}_k^{(i)} = \mathbf{m}_k + \sqrt{\mathbf{P}_k}\,\xi^{(i)}, \qquad i = 1, \ldots, 2n,$$

   where the unit sigma points are defined as

$$\xi^{(i)} = \begin{cases} \sqrt{n}\,\mathbf{e}_i &, \quad i = 1, \ldots, n \\ -\sqrt{n}\,\mathbf{e}_{i-n} &, \quad i = n+1, \ldots, 2n. \end{cases}$$

2. Propagate the sigma points through the dynamic model:

$$\hat{\mathcal{X}}_{k+1}^{(i)} = \mathbf{f}(\mathcal{X}_k^{(i)}), \quad i = 1, \ldots, 2n.$$

Cubature Rauch-Tung-Striebel Smoother (cont.)

3. Compute the predicted mean $\mathbf{m}_{k+1}^-$, the predicted covariance $\mathbf{P}_{k+1}^-$ and the cross-covariance $\mathbf{D}_{k+1}$:

$$
\begin{aligned}
\mathbf{m}_{k+1}^- &= \frac{1}{2n} \sum_{i=1}^{2n} \hat{\mathcal{X}}_{k+1}^{(i)} \\
\mathbf{P}_{k+1}^- &= \frac{1}{2n} \sum_{i=1}^{2n} (\hat{\mathcal{X}}_{k+1}^{(i)} - \mathbf{m}_{k+1}^-) \, (\hat{\mathcal{X}}_{k+1}^{(i)} - \mathbf{m}_{k+1}^-)^T + \mathbf{Q}_k \\
\mathbf{D}_{k+1} &= \frac{1}{2n} \sum_{i=1}^{2n} (\mathcal{X}_{k}^{(i)} - \mathbf{m}_k) \, (\hat{\mathcal{X}}_{k+1}^{(i)} - \mathbf{m}_{k+1}^-)^T.
\end{aligned}
$$

Cubature Rauch-Tung-Striebel Smoother (cont.)

4. Compute the gain $\mathbf{G}_k$, mean $\mathbf{m}_k^s$ and covariance $\mathbf{P}_k^s$ as follows:

$$
\begin{aligned}
\mathbf{G}_k &= \mathbf{D}_{k+1} \, [\mathbf{P}_{k+1}^-]^{-1} \\
\mathbf{m}_k^s &= \mathbf{m}_k + \mathbf{G}_k \, (\mathbf{m}_{k+1}^s - \mathbf{m}_{k+1}^-) \\
\mathbf{P}_k^s &= \mathbf{P}_k + \mathbf{G}_k \, (\mathbf{P}_{k+1}^s - \mathbf{P}_{k+1}^-) \, \mathbf{G}_k^T.
\end{aligned}
$$

Unscented Rauch-Tung-Striebel Smoother

1. Form the sigma points:

$$\begin{aligned}
\mathcal{X}_k^{(0)} &= \mathbf{m}_k, \\
\mathcal{X}_k^{(i)} &= \mathbf{m}_k + \sqrt{n+\lambda}\,\left[\sqrt{\mathbf{P}_k}\right]_i \\
\mathcal{X}_k^{(i+n)} &= \mathbf{m}_k - \sqrt{n+\lambda}\,\left[\sqrt{\mathbf{P}_k}\right]_i, \quad i = 1, \ldots, n.
\end{aligned}$$

2. Propagate the sigma points through the dynamic model:

$$\hat{\mathcal{X}}_{k+1}^{(i)} = \mathbf{f}(\mathcal{X}_k^{(i)}), \quad i = 0, \ldots, 2n.$$

Unscented Rauch-Tung-Striebel Smoother (cont.)

3. Compute predicted mean, covariance and cross-covariance:

$$
\begin{aligned}
\mathbf{m}_{k+1}^{-} &= \sum_{i=0}^{2n} W_i^{(m)}\, \hat{\mathcal{X}}_{k+1}^{(i)} \\
\mathbf{P}_{k+1}^{-} &= \sum_{i=0}^{2n} W_i^{(c)}\, (\hat{\mathcal{X}}_{k+1}^{(i)} - \mathbf{m}_{k+1}^{-})\, (\hat{\mathcal{X}}_{k+1}^{(i)} - \mathbf{m}_{k+1}^{-})^T + \mathbf{Q}_k \\
\mathbf{D}_{k+1} &= \sum_{i=0}^{2n} W_i^{(c)}\, (\mathcal{X}_{k}^{(i)} - \mathbf{m}_{k})\, (\hat{\mathcal{X}}_{k+1}^{(i)} - \mathbf{m}_{k+1}^{-})^T,
\end{aligned}
$$

Unscented Rauch-Tung-Striebel Smoother (cont.)

4. Compute gain smoothed mean and smoothed covariance: as follows:

$$
\begin{aligned}
\mathbf{G}_k &= \mathbf{D}_{k+1} \, [\mathbf{P}_{k+1}^-]^{-1} \\
\mathbf{m}_k^s &= \mathbf{m}_k + \mathbf{G}_k \, (\mathbf{m}_{k+1}^s - \mathbf{m}_{k+1}^-) \\
\mathbf{P}_k^s &= \mathbf{P}_k + \mathbf{G}_k \, (\mathbf{P}_{k+1}^s - \mathbf{P}_{k+1}^-) \, \mathbf{G}_k^T.
\end{aligned}
$$

Other Gaussian RTS Smoothers

- Gauss-Hermite RTS smoother is based on multidimensional Gauss-Hermite integration.
- Bayes-Hermite or Gaussian Process RTS smoother uses Gaussian process based quadrature (Bayes-Hermite).
- Monte Carlo integration based RTS smoothers.
- Central differences etc.

Particle Smoothing: Direct SIR

- The smoothing solution can be obtained from SIR by storing the whole state histories into the particles.
- Special care is needed on the resampling step.
- The smoothed distribution approximation is then of the form

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:T}) \approx \sum_{i=1}^{N} w_T^{(i)} \delta(\mathbf{x}_k - \mathbf{x}_k^{(i)}),$$

  where $\mathbf{x}_k^{(i)}$ is the kth component in $\mathbf{x}_{1:T}^{(i)}$.
- Unfortunately, the approximation is often quite degenerate.

- In backward-simulation particle smoother we simulate individual trajectories backwards.
- The simulated samples are drawn from the particle filter samples.
- Uses the previous filtering results in smoothing $\Rightarrow$ less degenerate than the direct SIR smoother.
- Idea:
  - Assume now that we have already simulated $\tilde{\mathbf{x}}_{k+1:T}$ from the smoothing distribution.
  - From the Bayesian smoothing equations we get

$$p(\mathbf{x}_k \mid \tilde{\mathbf{x}}_{k+1}, \mathbf{y}_{1:T}) \propto p(\tilde{\mathbf{x}}_{k+1} \mid \mathbf{x}_k)\, p(\mathbf{x}_k \mid \mathbf{y}_{1:k}).$$

Particle Smoothing: Backward Simulation [2/2]

Backward simulation particle smoother

Given the weighted set of particles $\{w_k^{(i)}, \mathbf{x}_k^{(i)}\}$ representing the filtering distributions:

- Choose $\tilde{\mathbf{x}}_T = \mathbf{x}_T^{(i)}$ with probability $w_T^{(i)}$.
- For $k = T-1, \ldots, 0$:
  1. Compute new weights by
  $$w_{k|k+1}^{(i)} \propto w_k^{(i)} p(\tilde{\mathbf{x}}_{k+1} \mid \mathbf{x}_k^{(i)})$$
  2. Choose $\tilde{\mathbf{x}}_k = \mathbf{x}_k^{(i)}$ with probability $w_{k|k+1}^{(i)}$

Given S iterations resulting in $\tilde{\mathbf{x}}_{1:T}^{(j)}$ for $j = 1, \ldots, S$ the smoothing distribution approximation is

$$p(\mathbf{x}_{1:T} \mid \mathbf{y}_{1:T}) \approx \frac{1}{S} \sum_j \delta(\mathbf{x}_{1:T} - \tilde{\mathbf{x}}_{1:T}^{(j)}).$$

Particle Smoothing: Reweighting [1/2]

- The reweighting particle smoother is based on computing new weights $w_{k+1|T}^{(i)}$ for the SIR filter particles such that:

$$p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T}) \approx \sum_i w_{k+1|T}^{(i)} \, \delta(\mathbf{x}_{k+1} - \mathbf{x}_{k+1}^{(i)}).$$

- Recall the smoothing equation

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:T}) = p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) \int \left[\frac{p(\mathbf{x}_{k+1} \mid \mathbf{x}_k) \, p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T})}{p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:k})} \right] \mathrm{d}\mathbf{x}_{k+1}$$

- We use SIR filter samples to form approximations (see booklet for details) as follows:

$$\int \frac{p(\mathbf{x}_{k+1} \mid \mathbf{x}_k) \, p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:T})}{p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:k})} \, d\mathbf{x}_{k+1} \approx \sum_i w_{k+1|T}^{(i)} \frac{p(\mathbf{x}_{k+1}^{(i)} \mid \mathbf{x}_k)}{p(\mathbf{x}_{k+1}^{(i)} \mid \mathbf{y}_{1:k})}$$

$$p(\mathbf{x}_{k+1} \mid \mathbf{y}_{1:k}) \approx \sum_j w_k^{(j)} \, p(\mathbf{x}_{k+1} \mid \mathbf{x}_k^{(j)})$$

Particle Smoothing: Reweighting [2/2]

Reweighting Particle Smoother

Given the weighted set of particles $\{w_k^{(i)}, x_k^{(i)}\}$ representing the filtering distribution, we can form approximations to the marginal smoothing distributions as follows:

- Start by setting $w_{T|T}^{(i)} = w_T^{(i)}$ for $i = 1, \ldots, n$.
- For each $k = T-1, \ldots, 0$ do the following:
  - Compute new importance weights by

$$w_{k|T}^{(i)} \propto \sum_j w_{k+1|T}^{(j)} \frac{w_k^{(i)} \, p(\mathbf{x}_{k+1}^{(j)} \mid \mathbf{x}_k^{(i)})}{\left[\sum_l w_k^{(l)} \, p(\mathbf{x}_{k+1}^{(j)} \mid \mathbf{x}_k^{(l)})\right]}.$$

At each step k the marginal smoothing distribution can be approximated as

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:T}) \approx \sum_i w_{k|T}^{(i)} \, \delta(\mathbf{x}_k - \mathbf{x}_k^{(i)}).$$

Rao-Blackwellized Particle Smoothing: Direct SIR

- Recall the Rao-Blackwellized particle filtering model:

$$\begin{aligned}
\mathbf{u}_k &\sim p(\mathbf{u}_k \mid \mathbf{u}_{k-1}) \\
\mathbf{x}_k &= \mathbf{A}(\mathbf{u}_{k-1})\,\mathbf{x}_{k-1} + \mathbf{q}_k, \qquad \mathbf{q}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{Q}) \\
\mathbf{y}_k &= \mathbf{H}(\mathbf{u}_k)\,\mathbf{x}_k + \mathbf{r}_k, \qquad \mathbf{r}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{R})
\end{aligned}$$

- The direct SIR based Rao-Blackwellized particle smoother:
  1. During filtering store the whole sampled state and Kalman filter histories to the particles.
  2. At the smoothing step, apply Rauch-Tung-Striebel smoothers to each of the Kalman filter histories.
- The smoothing distribution approximation:

$$p(\mathbf{x}_k, \mathbf{u}_k \mid \mathbf{y}_{1:T}) \approx \sum_{i=1}^{N} w_T^{(i)}\, \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)})\; \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k^{s,(i)}, \mathbf{P}_k^{s,(i)}).$$

- Also has the degeneracy problem.

Rao-Blackwellized Particle Smoothing: Other Types

- The RB backward-sampling smoother can be implemented in many ways:
  - Sample both the components backwards (leads to a pure sample representation).
  - Sample the latent variables only – requires quite complicated backward Kalman filtering computations.
  - Kim's approximation: just use the plain backward-sampling to the latent variable marginal.
- The RB reweighting particle smoothing is not possible exactly, but can be approximated using the above ideas.

Summary

- Bayesian smoothing is used for computing estimates of state trajectories given the measurements on the whole trajectory.
- Rauch-Tung-Striebel (RTS) smoother is the closed form smoother for linear Gaussian models.
- Extended, statistically linearized and unscented RTS smoothers are the approximate nonlinear smoothers corresponding to EKF, SLF and UKF.
- Gaussian RTS smoothers: cubature RTS smoother, Gauss-Hermite RTS smoothers and various others
- Particle smoothing can be done by storing the whole state histories in SIR algorithm.
- Rao-Blackwellized particle smoother is a combination of particle smoothing and RTS smoothing.

Matlab Demo: Pendulum [1/2]

- Pendulum model:

$$\begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_{k-1}^1 + x_{k-1}^2 \, \Delta t \\ x_{k-1}^2 - g \, \sin(x_{k-1}^1) \, \Delta t \end{pmatrix}}_{\mathbf{f(x_{k-1})}} + \begin{pmatrix} 0 \\ q_{k-1} \end{pmatrix}$$

$$y_k = \underbrace{\sin(x_k^1)}_{\mathbf{h}(\mathbf{x_k})} + r_k,$$

- The required Jacobian matrix for ERTSS:

$$\mathbf{F}_x(\mathbf{x}) = \begin{pmatrix} 1 & \Delta t \\ -g \, \cos(x^1) \, \Delta t & 1 \end{pmatrix}$$

Matlab Demo: Pendulum [2/2]

- The required expected value for SLRTSS is

$$\mathrm{E}[\mathbf{f}(\mathbf{x})] = \begin{pmatrix} m_1 + m_2\,\Delta t \\ m_2 - g\,\sin(m_1)\,\exp(-P_{11}/2)\,\Delta t \end{pmatrix}$$

- And the cross term:

$$\mathrm{E}[\mathbf{f}(\mathbf{x})\,(\mathbf{x}-\mathbf{m})^T] = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where

$$\begin{aligned}
c_{11} &= P_{11} + \Delta t\,P_{12} \\
c_{12} &= P_{12} + \Delta t\,P_{22} \\
c_{21} &= P_{12} - g\,\Delta t\,\cos(m_1)\,P_{11}\,\exp(-P_{11}/2) \\
c_{22} &= P_{22} - g\,\Delta t\,\cos(m_1)\,P_{12}\,\exp(-P_{11}/2)
\end{aligned}$$