Bounds in the Projective Unitary Group with Respect to Global Phase Invariant Metric

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***Abstract*—We consider a global phase-invariant metric in the projective unitary group $\mathcal{PU}_n$, relevant for universal quantum computing. We obtain the volume and measure of small metric ball in $\mathcal{PU}_n$ and derive the Gilbert-Varshamov and Hamming bounds in $\mathcal{PU}_n$. In addition, we provide upper and lower bounds for the kissing radius of the codebooks in $\mathcal{PU}_n$ as a function of the minimum distance. Using the lower bound of the kissing radius, we find a tight Hamming bound. Also, we establish bounds on the distortion-rate function for quantizing a source uniformly distributed over $\mathcal{PU}_n$. As example codebooks in $\mathcal{PU}_n$, we consider the projective Pauli and Clifford groups, as well as the projective group of diagonal gates in the Clifford hierarchy, and find their minimum distances. For any code in $\mathcal{PU}_n$ with given cardinality we provide a lower bound of covering radius. Also, we provide expected value of the covering radius of randomly distributed points on $\mathcal{PU}_n$, when cardinality of code is sufficiently large. We discuss codebooks at various stages of the projective Clifford + T and projective Clifford + S constructions in $\mathcal{PU}_2$, and obtain their minimum distance, distortion, and covering radius. Finally, we verify the analytical results by simulation.**

***Index Terms*—Projective Unitary Group, Volume, Kissing Radius, Hamming bound, and Gilbert-Varshamov bound .**

I. Introduction

In quantum computing, the design of quantum algorithms can be seen as a decomposition of a unitary matrix using a set of universal gates. It is well known that the set of Clifford gates combined with a non-Clifford gate forms a set of universal gates for quantum computation [1]. Exact decomposition or approximation of an arbitrary unitary matrix using a set of universal gates has been addressed in [2]–[4]. In [2], the total number of single-qubit gates that can be represented by the Clifford+T gates is calculated. An algorithm for finding a T-optimal approximation of single-qubit Z-rotations using Clifford+T gates is proposed in [3], which is capable of handling errors down to 10^{-15}. Approximating an arbitrary single-qubit gate from the special unitary group using Clifford+T gates, up to any given error threshold, is proposed in [4].

In quantum computation, the overall phase is irrelevant since it does not affect the measurable properties of a quantum system [1]. Hence, the gate approximation should be considered in the projective unitary group $\mathcal{PU}_n$ rather than in the unitary group or the special unitary group. $\mathcal{PU}_n$ consists of the equivalence classes of $n \times n$ unitary operations that differ by a global phase [5]. This makes the projective unitary group fundamental for constructing reliable quantum gates and enabling universal quantum computation.

A global phase invariant metric, which is suitable for $\mathcal{PU}_n$, is considered in [3], [6]–[8]. In [6], this metric is used for constructing the optimal fault-tolerant approximation of arbitrary gates with a set of discrete universal gates. Using this metric, the error approximations of universal gates are discussed in [7]. Furthermore, the $\mathbf{T}$-count and $\mathbf{T}$-depth of any multi-qubit unitary, which are crucial for optimizing quantum circuits, are analyzed in [8].

The volume of a small ball is needed for deriving the bounds on packing and covering problems. The volume of a small ball in the unitary group, Grassmannian, and Stiefel manifolds are well understood [9]–[13]. However, $\mathcal{PU}_n$ remains largely unexplored in the literature, particularly in terms of volume analysis and theoretical bounds.

The kissing radius, analogous to the packing radius in linear codes [14], plays a pivotal role in various applications, including the optimization of sphere-decoder algorithms [15], [16]. Also, the kissing radius relates to rate-distortion theory as it is the smallest possible distance from a codeword to the border of its Voronoi cell discussed in [17], [18]. Based on the volume of ball in the Grassmannian manifold, several bounds are derived for the rate–distortion tradeoff assuming that the cardinality of codebooks is sufficiently large [12].

Motivated by this, we consider the global phase-invariant metric in $\mathcal{PU}_n$ and compute the volume of a small ball. Using this volume, we derive the Hamming upper and Gilbert-Varshamov (GV) lower bounds. In addition, we obtain upper and lower bounds for the kissing radius as a function of the minimum distance of the codebook in $\mathcal{PU}_n$, and establish a tight Hamming bound. We derive upper and lower bounds for the distortion rate function. Furthermore, as examples of codebooks in $\mathcal{PU}_n$, we consider the projective Pauli group, the projective Clifford group, and the group of projective diagonal gates in the Clifford hierarchy, and determine their minimum distances. Finally, through the numerical results, we show the validity of our analyses.

The rest of this paper is organized as follows: Section II provides preliminaries. We derive the volume of metric balls for $\mathcal{PU}_n$ in Section III, and give the Hamming upper and GV lower bounds. Section IV provides the upper and lower bounds for the kissing radius as a function of the minimum distance.

Also, we obtain bounds on the distortion-rate function, lower bound of covering radius and approximated value of covering radius in $\mathcal{PU}_n$. Section VI discusses the simulation results, and Section VII concludes the paper.

II. Preliminaries

A. The Projective Unitary Group

The projective unitary group $\mathcal{PU}_n$ is a group of $n \times n$ complex valued matrices which can be represented in the quotient geometry as $\mathcal{U}_n/\mathcal{U}_1$, where $\mathcal{U}_n$ denotes the unitary group. The dimension of $\mathcal{PU}_n$ is n^2-1, and the elements are equivalence classes:

$$\mathcal{PU}_n = \{\alpha \mathbf{U} \;\mid\; \mathbf{U} \in \mathcal{U}_n \;\text{ and }\; |\alpha| = 1\}, \tag{1}$$

which can be represented by any unitary matrix $\mathbf{U}$ belonging to the class.

In this paper, we use the following metric [6]:

$$d(\mathbf{U}, \mathbf{V}) = \sqrt{1 - \frac{1}{n}|\mathrm{Tr}\,(\mathbf{U}^H\mathbf{V})|}, \tag{2}$$

for $\mathbf{U}, \mathbf{V} \in \mathcal{PU}_n$, where $(.)^H$ denotes the Hermitian conjugate. This is a metric on $\mathcal{PU}_n$, as it does not depend on the overall phase of the representation $\mathbf{U}$ of an element in $\mathcal{PU}_n$.

In [1], the relationship between the operator norm $d_O(\mathbf{U}, \mathbf{V}) = \max_\psi \|(\mathbf{U}-\mathbf{V})|\boldsymbol{\psi}\rangle\|$, where the maximum is over all pure states $|\boldsymbol{\psi}\rangle$, and the trace distance $\mathrm{Tr}\left(\sqrt{(\mathbf{U}-\mathbf{V})^H(\mathbf{U}-\mathbf{V})}\right)$ for single-qubit rotations is discussed, particularly in the context of approximating unitary operators. In determining these distances, the global phase of a unitary matrix plays a significant role. For example, for both of these metrics, the distance between $\mathbf{U}$ and $-\mathbf{U}$ is maximal, while for (2), their distance is zero, as they come from the same equivalence class. The phase invariant metric provides a notable advantage in finding optimal approximations, as it is invariant under global phase shifts.

B. Packing and Covering Problems

The packing problem is to fit a maximal set of non-overlapping balls of a given radius into the space. A finite subset of points in manifold $\mathcal{M}$

$$\mathcal{C} = \{\mathbf{C}_1, \ldots, \mathbf{C}_{|\mathcal{C}|}\} \subset \mathcal{M}, \tag{3}$$

is a $(|\mathcal{C}|, \delta)$-code, with

$$\delta = \min\{d(\mathbf{C}_i, \mathbf{C}_j) : \mathbf{C}_i, \mathbf{C}_j \in \mathcal{C}, i \neq j\} \tag{4}$$

the minimum distance.

The standard Hamming bound is a packing bound that provides an upper bound for the cardinality of a code given its minimum distance [13].

In non-Euclidean geometry, the maximum radius of the non-overlapping balls, known as the kissing radius, may be larger than half of the minimum distance. The kissing radius of a code $\mathcal{C}$ is defined as

$$\varrho = \sup_{\substack{B_{\mathbf{C}_l}(R) \cap B_{\mathbf{C}_k}(R) = \emptyset \\ \forall (k,l), k \neq l}} R, \tag{5}$$

where

$$B_{\mathbf{C}_i}(R) = \{\mathbf{P} \in \mathcal{M} \,:\, d(\mathbf{P}, \mathbf{C}_i) \leq R\} \tag{6}$$

is the metric ball of radius r centered on the codeword $\mathbf{C}_i$.

The covering problem is to find the minimum number of overlapping balls of a given radius, required to cover the entire space. The GV bound is a covering bound that provides a lower bound on the cardinality of the code, given its minimum distance [19].

C. Codebooks in the Projective Unitary Group

In this section, we consider three different codebooks in $\mathcal{PU}_n$: the projective Pauli group, the projective Clifford group, and the diagonal part of the Clifford Hierarchy. These codebooks play a crucial role in quantum theory [20]–[23]. In the following, we consider unitary matrices of dimension $n = 2^m$, with $m = 1, 2, \ldots$.

1) Projective Pauli Group : The 2×2 Pauli matrices are:

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The n-dimensional Pauli group is then defined as the set

$$\mathcal{P}_n = \{\pm \mathbf{D}(\mathbf{a}, \mathbf{b}), \pm i\mathbf{D}(\mathbf{a}, \mathbf{b})\},$$

where

$$\mathbf{D}(\mathbf{a}, \mathbf{b}) = \mathbf{X}^{a_1}\mathbf{Z}^{b_1} \otimes \mathbf{X}^{a_2}\mathbf{Z}^{b_2} \otimes \cdots \otimes \mathbf{X}^{a_m}\mathbf{Z}^{b_m},$$

with binary vectors $\mathbf{a} = [a_1, \ldots, a_m]^T, \mathbf{b} = [b_1, \ldots, b_m]^T$.

Moreover, $n = 2^m$-dimensional Heisenberg-Weyl group is defined as [24]

$$\mathcal{HW}_n = \{i^\kappa \mathbf{D}\left(\mathbf{a}, \mathbf{b}\right) | \mathbf{a}, \mathbf{b} \in \mathbb{F}_2^m, \kappa \in \mathbb{Z}_4\}, \tag{7}$$

where

$$\mathbb{Z}_{2^{k+1}} \triangleq \{e^{\frac{2\pi i}{2^k}q}\mathbf{I}_n \mid q = 0, 1, \ldots, 2^k - 1\}. \tag{8}$$

Also, we define

$$\mathbf{E}(\mathbf{a}, \mathbf{b}) = i^{a^T b}\mathbf{D}(\mathbf{a}, \mathbf{b}) \equiv \mathbf{E}(\mathbf{c}), \tag{9}$$

where $\mathbf{c} = [\mathbf{a}, \mathbf{b}] \in \mathbb{F}_2^{2m}$. We shall also use the normalized versions:

$$\tilde{\mathbf{E}}(\mathbf{c}) = \frac{1}{\sqrt{n}}\mathbf{E}(\mathbf{c}), \tag{10}$$

which form an orthonormal basis of the n^2-dimensional complex vector space of $n \times n$ complex matrices. The trace of these matrices satisfies:

$$\mathrm{Tr}(\mathbf{E}(\mathbf{c})) = 0, \quad \text{if } \mathbf{c} \neq 0,$$

so that $\mathbf{E}(\mathbf{0}) = \mathbf{I}_n$ is only basis matrix with non-vanishing trace, where $\mathbf{I}_n$ denotes $n \times n$ identity matrix. From the anti-commutation relation, it follows that the elements of the

Heisenberg-Weyl group either commute or anti-commute, and they are closed under multiplication:

$$\mathbf{E}(\mathbf{c})\mathbf{E}(\mathbf{c}') = \pm\mathbf{E}(\mathbf{c}')\mathbf{E}(\mathbf{c}) = \pm i\mathbf{E}(\mathbf{c}+\mathbf{c}').$$

The projective Pauli group is defined as $\widetilde{\mathcal{P}}_n = \mathcal{P}_n/\mathbb{Z}_4$, where the center is $\mathbb{Z}_{2^{k+1}}$ given by (8). Note that $\widetilde{\mathcal{P}}_n$ has cardinality $n^2 = 2^{2m}$.

2) Projective Clifford Group: The second level of the Clifford hierarchy is the Clifford group which is defined as

$$\mathcal{G}_n = \{\mathbf{G} \in \mathcal{U}_n \mid \mathbf{G}^H \mathcal{P}_n \mathbf{G} \subset \mathcal{P}_n\}. \tag{11}$$

The group of unitary automorphisms and unitary inner automorphisms of $\mathcal{P}_n$ is $\mathcal{G}_n$ and $\widetilde{\mathcal{P}}_n$, respectively and the unitary outer automorphism group are given by [21], [25]

$$\mathcal{G}_n/\widetilde{\mathcal{P}}_n = \mathrm{Sp}(2m, 2),$$

the binary symplectic group. This is the group of all binary $2m \times 2m$ matrices that fulfill:

$$\mathbf{F}\mathbf{\Omega}\mathbf{F}^T = \mathbf{\Omega}, \quad \text{where} \quad \mathbf{\Omega} = \begin{bmatrix} \mathbf{0}_m & \mathbf{I}_m \\ \mathbf{I}_m & \mathbf{0}_m \end{bmatrix}.$$

The isomorphism between the outer automorphisms and the symplectic group takes the form:

$$\mathbf{G_F}^H \mathbf{E}(\mathbf{c})\mathbf{G_F} = \pm\mathbf{E}(\mathbf{F}(\mathbf{c})), \tag{12}$$

i.e., for each symplectic binary matrix $\mathbf{F}$, there exists a unitary transform $\mathbf{G_F}$ which takes the Heisenberg-Weyl element corresponding to the binary vector $\mathbf{c}$ to the element corresponding to $\mathbf{F}(\mathbf{c})$, up to a sign. The sign is determined by the multiplications in $\mathcal{P}_n$ there are 2^m binary degrees of freedom, corresponding to the inner automorphisms. Explicit details on this can be found in [20]. The identity element in the automorphism group corresponds to the identity in the symplectic group:

$$\mathbf{G}_{\mathbf{I}_{2m}} = \mathbf{I}_n. \tag{13}$$

The projective Clifford group is defined as $\tilde{\mathcal{G}}_n = \mathcal{G}_n/\mathbb{Z}_8$, where $\mathbb{Z}_8$ is according to (8). The cardinality of $\tilde{\mathcal{G}}_n$ is given by [26]

$$\left|\tilde{\mathcal{G}}_n\right| = 2^{m^2+2m}\prod_{i=1}^{m}(2^{2i}-1). \tag{14}$$

3) Coodebooks from Higher Level of the Clifford Hierarchy: **Projective diagonal part of the Clifford hierarchy:**

The diagonal Clifford hierarchy of level k denoted by $\mathcal{D}_{n,k}$ forms a group and can be generated by the rotations of $\mathbf{Z}_j[\frac{\pi}{2^k}] = \exp\left(\frac{i\pi}{2^k}\mathbf{Z}_j\right)$, where $\mathbf{Z}_j$ is the Pauli $\mathbf{Z}$ acting on the jth qubit [27]. In particular, for m-qubit quantum system with $m \geq k$, the following gates construct $\mathcal{D}_{n,k}$

$$\left\langle \mathbf{Z}_i\left[\frac{\pi}{2^k}\right], \mathbf{\Lambda}^1_{i_1,i_2}\left(\mathbf{Z}\left[\frac{\pi}{2^{k-1}}\right]\right), \ldots, \mathbf{\Lambda}^{k-1}_{i_1,\ldots,i_k}\left(\mathbf{Z}[\tfrac{\pi}{2}]\right)\right\rangle,$$

where $\mathbf{\Lambda}^k(\mathbf{U})$ denotes the k-controlled $\mathbf{U}$ gate, acting on $k+1$ qubits. Here, i, $i_1, i_2, \ldots, i_k$ run over all qubits. For $m < k$ a similar set of generating gates truncated at $m' = m-1$ control gates construct $\mathcal{D}_{n,k}$. The projective diagonal part of the Clifford hierarchy is defined as $\tilde{\mathcal{D}}_{n,k} = \mathcal{D}_{n,k}/\mathbb{Z}_{2^k}$. The cardinality of $\tilde{\mathcal{D}}_{n,k}$ is given by [27]

$$|\tilde{\mathcal{D}}_{n,k}| = \prod_{j=0}^{\min(k-1,m-1)} \left(2^{k-j}\right)^{\binom{m}{j+1}}. \tag{15}$$

Projective Semi-Clifford codebook: The semi-Clifford $\mathcal{C}_{n,k}$ codebook is defined as a collection of unitary matrices $\mathbf{U}$ in such that it can be expressed as $\mathbf{U} = \mathbf{GDG}$, where $\mathbf{G} \in \mathcal{G}_n$ and $\mathbf{D} \in \mathcal{D}_{n,k}$ [24]. Similarly, we can define semi-Clifford projective codebook using the projective Clifford group and the projective diagonal part of the Clifford hierarchy and denote it by $\tilde{\mathcal{C}}_{n,k}$. In this paper, we are interested in single-qubit operations. In this case, the cardinality of $\tilde{\mathcal{C}}_{2,k}$ is given by

$$\left|\tilde{\mathcal{C}}_{2,k}\right| = 24\left(3\cdot 2^{k-2}-2\right) \tag{16}$$

Codebooks of products of 3rd and 4th level Clifford hierarchy elements: In [2] single-qubit gate operations were approximated using codebooks of products of Clifford+$\mathbf{T}$-gates, i.e. products of 3rd level Clifford hierarchy elements. We define a codebook given by the maximal number l of $\mathbf{T}$-gates in the codewords. The cardinality of the code considered in [2] is then $192\left(3\cdot 2^l-2\right)$, where 192 is the cardinality of the single-qubit Clifford gates. We denote by $\tilde{\mathcal{C}}^l_{2,3}$ the restriction of this codebook to the projective unitary group $\mathcal{PU}_2$. It follows that the cardinality of the codebook with at most l $\mathbf{T}$-gates is

$$|\tilde{\mathcal{C}}^l_{2,3}| = 24\left(3\cdot 2^l-2\right). \tag{17}$$

Motivated by this codebook, we also consider using a similar codebook with at most l $\mathbf{S} = \sqrt{\mathbf{T}}$ gates, which means that the codebook consists of products of elements in the 4th level of the Clifford hierarchy. We denote this codebook by $\tilde{\mathcal{C}}^l_{2,4}$. Using a similar argument as in [2], the cardinality of the codebook becomes

$$|\tilde{\mathcal{C}}^l_{2,4}| = 24\left(\frac{9\left(6^l-1\right)}{5}+1\right). \tag{18}$$

III. Volume of Projective Unitary Group

In this section, we find the volume of the $\mathcal{PU}_n$ and using this volume, we derive a measure of small metric ball in $\mathcal{PU}_n$ with respect to the metric given by (2). In addition, we provide the Hamming upper and GV lower bounds in $\mathcal{PU}_n$.

The Euclidean $(D-1)$-sphere of radius R in $\mathbb{R}^D$ is defined as $\mathcal{S}^{D-1}(R) = \left\{\mathbf{x} \in \mathbb{R}^D \,\middle|\, \|\mathbf{x}\|_2 = R\right\}$. The volume of $\mathcal{S}^{D-1}(R)$ is given by

$$\mathrm{V}_D(R) = \frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}+1\right)}R^D. \tag{19}$$

Also, the volume of the unitary group $\mathcal{U}_n$ is [13]

$$\mathrm{Vol}(\mathcal{U}_n) = \frac{(2\pi)^{\frac{n(n+1)}{2}}}{\prod_{i=1}^{n}(i-1)!}. \tag{20}$$

Theorem 1. *The volume of the $\mathcal{PU}_n$ is*

$$\mathrm{Vol}\left(\mathcal{PU}_n\right) = \frac{(2\pi)^{\frac{n(n+1)}{2}}}{2\pi\sqrt{n}\prod_{i=1}^{n}(i-1)!}. \tag{21}$$

Proof. According to the [5], the volume of a homogeneous quotient space $\mathcal{G}/\mathcal{K}$ arising from the free and proper action of subgroup $\mathcal{K}$ on group $\mathcal{G}$ is $\mathrm{Vol}\left(\mathcal{G}\right)/\mathrm{Vol}\left(\mathcal{K}\right)$. The volume of $\mathrm{Vol}(\mathcal{U}_n)$ is given by (20). The subgroup forming the cosets in (1) is isomorphic to $\mathcal{U}_1$, but strictly speaking not isometric. To find a metric on this subgroup, consider $\mathbf{X} = e^{i\theta}\mathbf{I}_n$ and $\mathbf{X}' = \mathbf{X} + d\mathbf{X}$ where $d\mathbf{X} = ie^{i\theta}\mathbf{I}_n\, d\theta$. The infinitesimal distance is given by

$$(ds)^2_{\mathcal{U}_1} = \|\mathbf{X} - \mathbf{X}'\|_F^2 = \|d\mathbf{X}\|_F^2 = n\, d^2\theta. \tag{22}$$

Therefore, the subgroup divided away is isometric to a circle with radius $\sqrt{n}$, and the volume of subgroup $\mathcal{K}$ in $\mathcal{U}_n$ is $\mathrm{Vol}(\mathcal{K}) = \int_0^{2\pi}\sqrt{n}d\theta = 2\pi\sqrt{n}$. The statement follows directly. ☐

The measure of the metric ball in the manifold $\mathcal{M}$, considering the Frobenius norm, is defined as

$$\mu_{\mathrm{F}}\left(B(R)\right) = \frac{\mathrm{Vol}\left(B(R)\right)}{\mathrm{Vol}(\mathcal{M})}, \tag{23}$$

where $\mathrm{Vol}\left(B(R)\right)$ is the volume of the ball with radius R in the manifold. For the measure of the metric ball in $\mathcal{PU}_n$ we have

Corollary 1. *As $R \to 0$, the measure of a metric ball $B(R)$ in $\mathcal{PU}_n$ with respect to the global phase invariant metric* (2) *is*

$$\mu_{\mathrm{d}}(B(R)) = c_n\, R^D(1 + \mathcal{O}(R^2)) \tag{24}$$

where $c_n = \frac{(2\pi)^{-\frac{(n-1)}{2}}\, n^{\frac{n^2}{2}}}{\Gamma(\frac{n^2-1}{2}+1)}\prod_{i=1}^{n}(i-1)!$, *and* $D = n^2 - 1$ *is the dimension of* $\mathcal{PU}_n$.

Proof. The measure of metric ball in $\mathcal{PU}_n$ with respect to the metric (2) can be written as

$$\begin{aligned} F_d\left(R\right) &= \Pr\{d \le R\} = \Pr\left\{\frac{d_F}{\sqrt{2n}} \le R\right\} \\ &= \mu_{\mathrm{F}}\left(B\left(\sqrt{2n}R\right)\right), \end{aligned} \tag{25}$$

where d_{F} denotes the Frobenius distance. The volume of a small ball can be well approximated by the volume of a ball of equal radius in the tangent space [13] as

$$\mathrm{Vol}\left(B(R)\right) = \mathrm{V}_D(R)\left(1 + O(R^2)\right). \tag{26}$$

Substituting (26) and (21) in (23) and considering (25) completes the proof. ☐

IV. Minimum-distance Bounds on $\mathcal{PU}_n$

The GV and Hamming bounds provide lower and upper bounds on the cardinality of a codebook in the manifold [9]. In the following, we provide these bounds for $\mathcal{PU}_n$. There exists a codebook $\mathcal{C}$ in $\mathcal{PU}_n$ with cardinality $|\mathcal{C}|$ and the minimum distance δ with respect to the metric (2) such that

$$\frac{1}{\mu_d(B(\delta))} \le |\mathcal{C}|. \tag{27}$$

Also, for any $(|\mathcal{C}|, \delta)$-codebook in $\mathcal{PU}_n$

$$|\mathcal{C}| \le \frac{1}{\mu_d(B(\frac{\delta}{2}))}. \tag{28}$$

The Hamming bound is a packing bound, literally bounding the number of codewords surrounded by $B(\frac{\delta}{2}))$-balls that can be packed into the manifold.

The Gilbert-Varhamov bound arises from a covering argument. If $|\mathcal{C}|$ balls $B(\delta)$ do not cover the manifold, there is room to add one more point which is at least at distance δ from all other points.

The Hamming bound can be enhanced by analyzing the kissing radius.

A. Kissing Radius Bounds of Projective Unitary Group

In this section, we derive upper and lower bounds for the kissing radius ϱ as a function of the minimum distance of a code in $\mathcal{PU}_n$ with respect to the global phase-invariant metric. Moreover, we establish a tight Hamming bound in this context, using the density $\Delta(\mathcal{C})$ of a code. For a code with cardinality K and kissing radius ϱ, the density is [13]:

$$\Delta(\mathcal{C}) = K\mu_d(B(\varrho)).$$

Lemma 1. *Let $\mathbf{U}, \mathbf{V} \in \mathcal{PU}_n$, and define $\mathbf{W} = \mathbf{U}^H\mathbf{V}$. Considering the metric* (2)*, the geodesics midpoint between $\mathbf{U}$ and $\mathbf{V}$ is given by*

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Omega}\sqrt{\mathbf{L}}\boldsymbol{\Omega}^H, \tag{29}$$

where $\mathbf{W} = \boldsymbol{\Omega}\mathbf{L}\boldsymbol{\Omega}^H$ with $\mathbf{L} = \mathrm{diag}\left(e^{j\theta_1}, ..., e^{j\theta_n}\right)$.

Proof. Since $\mathcal{PU}_n$ is a Lie group, its geodesic can be described using its Lie algebra $\mathfrak{pu}(n)$. As discussed in [28], the Lie algebra of $\mathcal{PU}_n$ is $\mathfrak{pu}(n) \cong \mathfrak{u}(n)/\{ia\mathbf{I}\}$, where $\mathfrak{u}(n)$ is the Lie algebra of $\mathcal{U}_n$. The geodesic curve in $\mathcal{PU}_n$ is given by

$$\boldsymbol{\gamma}(t) = \mathbf{U}e^{t(\mathbf{A}+ia\mathbf{I})} = \mathbf{U}e^{t\mathbf{A}'}, \qquad 0 \le t \le 1$$

where $\mathbf{A}, \mathbf{A}' \in \mathfrak{u}(n)$ is a skew-Hermitian matrix such that $\boldsymbol{\gamma}(0) = \mathbf{U}, \boldsymbol{\gamma}(1) = \mathbf{V} = \mathbf{U}e^{\mathbf{A}'}$. The geodesics midpoint is given by $\boldsymbol{\gamma}(1/2)$ which can be written in the form of $\mathbf{M}$ given in (29). Using (2), it follows that $d\left(\mathbf{U}, \mathbf{M}\right) = d\left(\mathbf{V}, \mathbf{M}\right)$. ☐

The kissing radius of a given code is hard to determine since it depends on the minimum distance of the code and principal angles between codewords [29]. The following theorem provides lower and upper bounds for the kissing radius of a code in $\mathcal{PU}_n$ with the help of Lemma 1. For the finding the upper and lower bound of kissing radius, we have to find the upper and lower bound of following expression: let

$$\varrho=\sqrt{1-\frac{1}{n}\Big|\sum_{j=1}^{n}e^{\frac{i\theta_j}{2}}\Big|}\quad\text{given that}\quad n(1-\delta^2)=\Big|\sum_{j=1}^{n}e^{i\theta_j}\Big|,\tag{30}$$

where $n=2^m$, $m=1,2,3,\cdots$. Let $\bar{\varrho}$ be an upper bound and $\underline{\varrho}$ be a lower bound on the kissing radius ϱ. Then, the following theorem gives bounds for the kissing radius:

Theorem 2. *For any code $(|\mathcal{C}|,\delta)\in\mathcal{PU}_n$, the kissing radius ϱ is bounded as*

$$\underline{\varrho}\le\varrho\le\bar{\varrho},$$

where $\underline{\varrho}=\sqrt{1-\sqrt{1-\frac{\delta^2}{2}}}$ and $\bar{\varrho}=\sqrt{1-\sqrt{\frac{1+(1-\delta^2)^2}{2}}}$. The corresponding bounds on codebook density are

$$|\mathcal{C}|\,\mu_d(B(\underline{\varrho}))\le\Delta(\mathcal{C})\le\min\{1,|\mathcal{C}|\,\mu_d(B(\bar{\varrho}))\},.\tag{31}$$

Proof. According to (30), we consider optimizing the kissing radius given the minimum distance. First we calculate upper bound of kissing radius in the $\mathcal{PU}_n$. We can see that

$$\Big|\sum_{j=1}^{n}e^{i\theta_j}\Big|^2=n+2\sum_{1\le i<j\le n}\cos(\theta_i-\theta_j),$$

and

$$\begin{aligned}\Big|\sum_{j=1}^{n}e^{\frac{i\theta_j}{2}}\Big|^2&=n+2\sum_{1\le i<j\le n}\cos\left(\frac{\theta_i-\theta_j}{2}\right)\\&=n+2\sum_{1\le i<j\le n}\sqrt{\frac{\cos(\theta_i-\theta_j)+1}{2}}.\end{aligned}\tag{32}$$

For $0\le x\le1$, we know $\sqrt{x}\ge x$. From this, it follows that

$$\begin{aligned}&n+2\sum_{1\le i<j\le n}\sqrt{\frac{\cos(\theta_i-\theta_j)+1}{2}}\\&\ge n+2\sum_{1\le i<j\le n}\left(\frac{\cos(\theta_i-\theta_j)+1}{2}\right)\\&=\frac{n^2+\Big|\sum\limits_{j=1}^{n}e^{i\theta_j}\Big|^2}{2}=\frac{n^2+n^2(1-\delta^2)^2}{2}\end{aligned}$$

From (32), we have an inequality

$$\Big|\sum_{j=1}^{n}e^{\frac{i\theta_j}{2}}\Big|\ge\sqrt{\frac{n^2+n^2(1-\delta^2)^2}{2}}.$$

Hence upper bound of kissing radius is:

$$\varrho\le\sqrt{1-\sqrt{\frac{1+(1-\delta^2)^2}{2}}}.$$

For lower bound, we need to find $\max\Big|\sum\limits_{j=1}^{n}e^{\frac{i\theta_j}{2}}\Big|$ such that $n(1-\delta^2)=\Big|\sum\limits_{j=1}^{n}e^{i\theta_j}\Big|$.

For $\max\Big|\sum\limits_{j=1}^{n}e^{\frac{i\theta_j}{2}}\Big|$ the triangle inequality states that

$$\left|\sum_{j=1}^{n}e^{\frac{i\theta_j}{2}}\right|\le\sum_{j=1}^{n}\left|e^{\frac{i\theta_j}{2}}\right|=n,$$

with equality if and only if all the vectors $e^{\frac{i\theta_j}{2}}$ are aligned, i.e., their arguments $\frac{\theta_j}{2}$ differ by a multiple of 2π. Similarly, the constraint $\left|\sum_{j=1}^{n}e^{i\theta_j}\right|=n(1-\delta^2)$ implies that the vectors $e^{i\theta_j}$ are not fully aligned unless $\delta=0$.

To maintain the constraint, we split the n angles into two groups. We assume $\frac{n}{2}-k$, θ_j would be same θ in one group and $\frac{n}{2}+k$, θ_j would be same ϕ in other group. These angles satisfy the constraint

$$n(1-\delta^2)=\left|\sum_{j=1}^{n}e^{i\theta_j}\right|=\left|\left(\frac{n}{2}-k\right)e^{i(\theta-\phi)}+\left(\frac{n}{2}+k\right)\right|$$

i.e.,

$$\big(n(1-\delta^2)\big)^2=\frac{2n^2}{4}+2k^2+2\left(\frac{n^2}{4}-k^2\right)\cos(\theta-\phi)$$

$$\frac{1}{2\left(\frac{n^2}{4}-k^2\right)}\left(\big(n(1-\delta^2)\big)^2-\frac{2n^2}{4}-2k^2\right)=\cos(\theta-\phi).$$

So we have

$$-1\le\frac{\big(n(1-\delta^2)\big)^2-\frac{2n^2}{4}-2k^2}{2\left(\frac{n^2}{4}-k^2\right)}\le1$$

if and only if $k=0$. Without loss of generality we set

$$\theta_1=\theta_2=\cdots=\theta_{\frac{n}{2}}=\theta\quad\text{and}\quad\theta_{\frac{n}{2}+1}=\theta_{\frac{n}{2}+2}=\cdots=\theta_n=\phi.$$

It follows that $n(1-\delta^2)=\Big|\sum\limits_{j=1}^{n}e^{i\theta_j}\Big|=\Big|\frac{n}{2}e^{i\theta}+\frac{n}{2}e^{i\phi}\Big|$, i.e., $4(1-\delta^2)^2=2+2\cos(\theta-\phi)$. This implies that

$$\cos(\theta-\phi)=2(1-\delta^2)^2-1.$$

Let us now examine,

$$\begin{aligned}\Big|\sum_{j=1}^{n}e^{\frac{i\theta_j}{2}}\Big|^2&=\Big|\frac{n}{2}e^{i\frac{\theta}{2}}+\frac{n}{2}e^{i\frac{\phi}{2}}\Big|^2\\&=\frac{n^2}{4}\left(2+2\sqrt{\frac{\cos(\theta-\phi)+1}{2}}\right)\\&=\frac{n^2}{4}\left(4-2\delta^2\right).\end{aligned}$$

So, $\Big|\sum\limits_{j=1}^{n}e^{\frac{i\theta_j}{2}}\Big|=n\sqrt{1-\frac{\delta^2}{2}}$. Thus, we obtain the lower bound for the kissing radius:

$$\varrho\ge\sqrt{1-\sqrt{1-\frac{\delta^2}{2}}}.$$

□

One of the central problems in coding theory is determining the maximum size of a codebook for a given minimum distance. Using the normalized volume of the metric ball $\mu_d(B(r))$, as given in Corollary 1, and applying Theorem 2, we obtain the following refined version of the Hamming bound:

Corollary 2. *For any $(|\mathcal{C}|, \delta)$-code in $\mathcal{PU}_n$, we have*

$$|\mathcal{C}| \leq \frac{1}{\mu_d(B(\underline{\varrho}))}, \tag{33}$$

where $\underline{\varrho}$ is given in Theorem 2.

Proof. From (31), $|\mathcal{C}|\, \mu_d(B(\underline{\varrho})) \leq \Delta(\mathcal{C}) \leq 1$, and we have $\frac{\delta}{2} \leq \underline{\varrho}$. It implies that

$$|\mathcal{C}| \leq \frac{1}{\mu_d(B(\underline{\varrho}))} \leq \frac{1}{\mu_d(B(\frac{\delta}{2}))}.$$

□

B. Minimum Distances of the Projective Unitary Codebooks

The minimum distance of quantum codebooks plays a pivotal role in facilitating both error correction and error detection [30]–[32]. Here, we find the minimum distance of the example codebooks in $\mathcal{PU}_n$.

Lemma 2. *For any $n \times n$ matrix $\mathbf{M}$ and an orthonormal basis $\{\mathbf{B}^{(n)}\}$ for the vector space $\mathcal{M}_{\mathbb{C}}(n)$, we have*

$$\sum_n \mathrm{Tr}\left(\mathbf{M}^H \mathbf{B}^{(n)H} \mathbf{M} \mathbf{B}^{(n)}\right) = |\mathrm{Tr}\,(\mathbf{M})|^2.$$

Proof. This follows directly from the completeness of the basis. The linear mapping from $\mathcal{M}_{\mathbb{C}}(n)$ to itself, given by the sum of the outer products of the basis elements with themselves, is the identity. Explicitly, this means

$$\sum_n b_{i,j}^{(n)*} b_{k,l}^{(n)} = \delta_{i,k}\delta_{j,l},$$

where $b_{i,j}^{(n)}$ are the matrix elements of $\mathbf{B}^{(n)}$, and $\delta_{i,k}$ is the Kronecker delta function. □

Lemma 3. *The inner products of the transformed Heisenberg-Weyl matrices with themselves take the values*

$$\mathrm{Tr}\left(\mathbf{G}_{\mathbf{F}}^H \tilde{\mathbf{E}}(\mathbf{c}) \mathbf{G}_{\mathbf{F}} \tilde{\mathbf{E}}(\mathbf{c})\right) = \begin{cases} \pm 1 & \text{if } \mathbf{F}(\mathbf{c}) + \mathbf{c} = 0 \mod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from (12), and the definition (9) of the Heisenberg-Weyl matrices. From these we see that the matrix within the trace is $\mathbf{E}(\mathbf{F}(\mathbf{c}) + \mathbf{c})/n$, up to an integer power of i. The trace is non-vanishing only if this $\mathbf{E}$ is proportional to identity. This occurs if and only if $\mathbf{F}(\mathbf{c}) + \mathbf{c} = 0 \mod 2$, and according to (9) the integer power of i determining the sign ± 1. □

Proposition 1. *The minimum distances of $\widetilde{\mathcal{P}}_n$ and $\tilde{\mathcal{G}}_n$ considering the global phase invariant metric are*

$$\delta_p = 1\,, \qquad \delta_c = \sqrt{1 - \frac{1}{\sqrt{2}}}\,, \tag{34}$$

respectively.

Proof. As $\widetilde{\mathcal{P}}_n$ is a group under multiplication, let $\mathbf{U}^H\mathbf{V} = \mathbf{W} = e^{\frac{2\pi i}{2^k}q}\mathbf{D}(\mathbf{a}, \mathbf{b}) \in \widetilde{\mathcal{P}}_n$, i.e., $d(\mathbf{U}, \mathbf{V}) = d(\mathbf{I}, \mathbf{W})$. Then $|\mathrm{Tr}\left(\mathbf{I}^H\mathbf{W}\right)| = |\mathrm{e}^{\frac{2\pi i}{2^k}q}\,\mathrm{Tr}\left(\mathbf{D}(\mathbf{a}, \mathbf{b}\right)|$. We have

$$|\mathrm{e}^{\frac{2\pi i}{2^k}q}\,\mathrm{Tr}\left(\mathbf{D}(\mathbf{a}, \mathbf{b}\right)| = \begin{cases} n, & \mathbf{a} = \mathbf{b} \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

From (2) and (35), the distance of any two codewords in $\widetilde{\mathcal{P}}_n$ is zero or 1. Hence, $\delta_p = 1$.

For finding δ_c, first we observe that

$$|\mathrm{Tr}(\mathbf{G}_{\mathbf{F}})| \leq \sqrt{\sum_{\mathbf{c}} \left|\mathrm{Tr}\left(\mathbf{G}_{\mathbf{F}}^H \tilde{\mathbf{E}}(\mathbf{c}) \mathbf{G}_{\mathbf{F}} \tilde{\mathbf{E}}(\mathbf{c})\right)\right|}. \tag{36}$$

This follows from Lemma 2, as the matrices $\tilde{\mathbf{E}}(\mathbf{c})$ form an orthonormal basis in $\mathcal{M}_{\mathbb{C}}(n)$, and from the triangle inequality. Using Lemma 3, a term in the sum over $\mathbf{c}$ contributes a factor of 1 if $\mathbf{F}(\mathbf{c}) + \mathbf{c} = 0 \mod 2$. The sum is over all binary $2m$-vectors, so the result is given by the number of vectors in the null space of $\mathbf{F} + \mathbf{I_{2m}}$, which evaluates to

$$n_0 = 2^{2m - \mathrm{rank}(\mathbf{F} + \mathbf{I_{2m}})}.$$

For $\mathbf{F} = \mathbf{I}_{2m}$, we have $n_0 = n^2$, corresponding to $\mathbf{G}_{\mathbf{F}} = \mathbf{I_n}$. Otherwise, for $\mathbf{G}_{\mathbf{F}} \neq \mathbf{I}_n$, $\mathrm{rank}(\mathbf{F} + \mathbf{I_{2m}}) \geq \mathbf{1}$, so $n_0 \leq 2^{2m-1}$. From the above inequality, we find $|\mathrm{Tr}(\mathbf{G}_{\mathbf{F}})| \leq \frac{n}{\sqrt{2}}$. Since the bound depends only on $|\mathbf{G}_{\mathbf{F}}|$, considering the center of $\tilde{\mathcal{G}}_n$ does not change the result. Using the equation (2) the minimum global phase invariant distance between two matrices in the $\tilde{\mathcal{G}}_n$ is

$$\delta_c^2 \geq \left(1 - \frac{2n}{n\sqrt{2}}\right).$$

Hence $\delta_c = \sqrt{1 - \frac{1}{\sqrt{2}}}$.

□

Proposition 2. *The minimum distance of $\tilde{\mathcal{D}}_{n,k}$ with respect to* (2) *is*

$$\delta_d = \sqrt{1 - \cos\left(\frac{\psi_k}{2}\right)}, \text{ where } \psi_k = \frac{2\pi}{2^k}.$$

Proof. As $\tilde{\mathcal{D}}_{n,k}$ is a group under multiplication, let $\mathbf{U}^H\mathbf{V} = \mathbf{W} \in \tilde{\mathcal{D}}_{n,k}$, i.e., $d(\mathbf{U}, \mathbf{V}) = d(\mathbf{I}, \mathbf{W})$. Then, using a similar approach as in [33], we can find the minimum distance. Basically, in an m-qubit system, setting $\mathbf{W} = \mathbf{I}^{\otimes(m-1)} \otimes \mathbf{Z}_m\left[\frac{\pi}{2^k}\right]$ in $d\left(\mathbf{I}, \mathbf{W}\right)$ results in the minimum distance. □

In general, determining the minimum distances of the semi-Clifford, $\tilde{\mathcal{C}}_{2,3}^l$, and $\tilde{\mathcal{C}}_{2,4}^l$ codebooks is nontrivial; hence, we employ numerical simulations to estimate their minimum distances.

V. BOUNDS ON CODEBOOK DISTORTION

In universal quantum computation, the goal is to approximate a given unitary gate by the closest element of a universal gate set. With a finite computational codebook, this inevitably leads to distortion—the executed circuit is only an approximation of the desired circuit. Accordingly, the average distortion, or the largest distortion may be of more interest for quantum computation than the minimum distances of the codebooks.

A. Distortion-Rate function in $\mathcal{PU}_n$

The quantization problem of approximating the target using the codebook of available circuits, is directly related to rate-distortion theory. The distortion rate function is defined as [12]

$$\mathcal{D}^*(K) = \inf_{\mathcal{C}:|\mathcal{C}|=K} \mathcal{D}(\mathcal{C}), \tag{37}$$

where

$$\mathcal{D}(\mathcal{C}) = \mathbb{E}\left[\min_{\mathbf{P}\in\mathcal{C}} d^2(\mathbf{P},\mathbf{Q})\right], \tag{38}$$

where $\mathcal{C}$ in $\mathcal{PU}_n$ with cardinality $|\mathcal{C}| = K$. Here, $\mathbf{Q}$ is an arbitrary point in the space.

Based on the volume of $\mathcal{PU}_n$ given in Corollary 1, the distortion–rate tradeoff is characterized by establishing lower and upper bounds on the distortion–rate function. For a codebook $\mathcal{C}$ with sufficiently large cardinality K, the distortion–rate function over the $\mathcal{PU}_n$, with global phase invariant distance, can be bounded as

$$\frac{D}{D+2}\left(c_n K\right)^{-\frac{2}{D}} \leq \mathcal{D}^*(K) \leq \frac{2\Gamma\left(\frac{2}{D}\right)}{D}\left(c_n K\right)^{-\frac{2}{D}}\left(1+o(1)\right), \tag{39}$$

where $D = n^2 - 1$ and c_n given in Corollary 1. This is an extension of the results in [12] for Grassmannian manifold to $\mathcal{PU}_n$.

As discussed in [12], for a code (K,δ) in $\mathcal{PU}_n$ the distortion is upper bounded as

$$\mathcal{D}(\mathcal{C}) \leq \left(\frac{\delta^2}{4} - 1\right) K\mu_d(B(\delta/2)) + 1\,. \tag{40}$$

Note that in a flat space, the packing radius is $\frac{\delta}{2}$. However, in a non-flat geometry $\frac{\delta}{2} \leq \varrho \leq \delta$. Therefore, for any code $(|\mathcal{C}|,\delta) \in \mathcal{PU}_n$ and using the lower bound of the kissing radius, we can have a tighter upper bound on the distortion than (40). Hence using (40) and Propositions 1 and 2, we can obtain distortion upper bounds for codebooks $\widetilde{\mathcal{P}}_n$, $\tilde{\mathcal{G}}_n$ and $\tilde{\mathcal{D}}_{n,k}$, $\tilde{\mathcal{C}}_{2,k}$, $\tilde{\mathcal{C}}^l_{2,3}$ and $\tilde{\mathcal{C}}^l_{2,4}$.

B. Covering radius

The worst-case disotrtion is governed by the covering radius of the codebook in $\mathcal{PU}_n$. With $\mathcal{C} = \{\mathbf{P}_1,\ldots,\mathbf{P}_K\}$ a codebook of K points over $\mathcal{PU}_n$. The covering radius ρ is

$$\rho = \max_{\mathbf{U}\in\mathcal{PU}_n} \min_{1\leq \mathbf{P}_i \leq K} d(P_i, U) \tag{41}$$

and it is thus the square root of the maximum distortion. A lower bound for the covering radius follows directly from a covering argument.

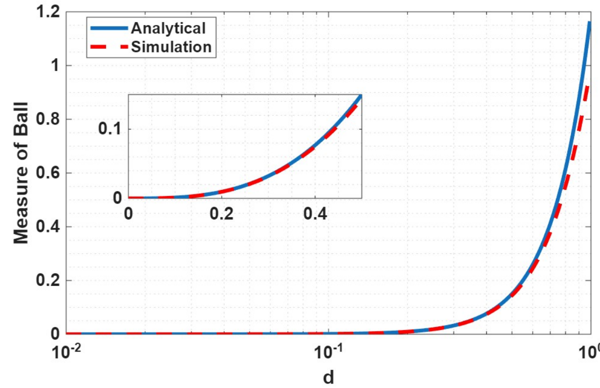


Fig. 1: Theoretical and simulation results comparison of the measure of the ball in $\mathcal{PU}_2$, given by Corollary 1.

As $\mathcal{PU}_n$ is a compact manifold, each open cover of $\mathcal{PU}_n$ has a finite sub-cover. Consider a ball $B_\rho(\mathbf{P}_i)$ centered at each codeword $\mathbf{P}_i \in \mathcal{C}$. By definition of the covering radius we have $\mathcal{PU}_n \subseteq \bigcup_{i=1}^K B_\rho(\mathbf{P}_i)$. This implies that $\mathrm{Vol}(\mathcal{M}) \leq K\mathrm{Vol}(B(\rho))$, i.e., $K\mu_d(B(\rho)) \geq 1$. Using Corollary 1, a lower bound of the covering radius is

$$\rho \geq \left(\frac{1}{c_n K}\right)^{1/D}. \tag{42}$$

The expected value of the covering radius of a random codebook consisting of K points selected uniformly at random from the Grassmannian manifold can be found in [34], while a general proof for the expected value of the covering radius of random codebooks on compact manifolds is provided in [35].

From these works, we find the expected value of the covering radius of a random codebook on $\mathcal{PU}_n$ with a sufficiently large cardinality K as follows: Let $\mathcal{C}_K = \{\mathbf{P}_1,\ldots,\mathbf{P}_K\}$ be a set of K points selected independently and uniformly at random from $\mathcal{PU}_n$ with respect to the measure μ_d. Then

$$\lim_{K\to\infty} \mathbb{E}\left(\rho\right)\left(\frac{K}{\log K}\right)^{\frac{1}{D}} = \left(\frac{\mathrm{Vol}(\mathcal{PU}_n)}{\mathrm{V}_D(1)}\right)^{\frac{1}{D}}. \tag{43}$$

The covering radius can be approximated as

$$\rho \approx \left(\frac{\mathrm{Vol}(\mathcal{PU}_n)}{\mathrm{V}_D(1)}\frac{\log K}{K}\right)^{\frac{1}{D}}. \tag{44}$$

VI. SIMULATION RESULTS

In this section, we verify the correctness of our analyses in $\mathcal{PU}_n$ using numerical results. First, in Fig. 1, we consider the measure of the ball in $\mathcal{PU}_n$ given by Corollary 1. This figure illustrates the small ball volume in $\mathcal{PU}_n$ for $n = 2$ in terms of the global phase invariant metric. The simulation results are obtained by averaging over 10^8 unitary matrices generated uniformly at random with the Haar measure, following [36]. Note that due to the quotient structure, this also provides the Haar measure in $\mathcal{PU}_n$. The simulations results for small values of the distance matches with the theoretical evaluation.

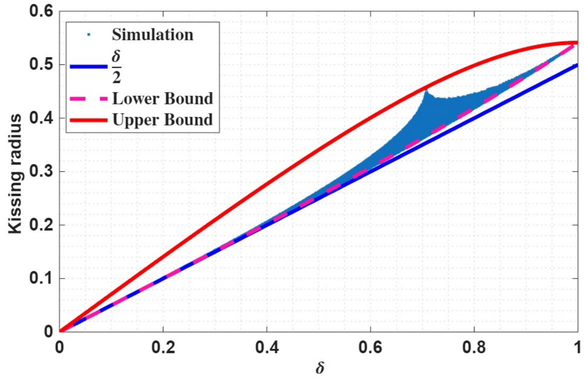


Fig. 2: The upper bound $\bar{\varrho}$ and lower bound $\underline{\varrho}$ of kissing radius in $\mathcal{PU}_4$. These bounds are compared to simulated midpoints between two randomly generated codewords

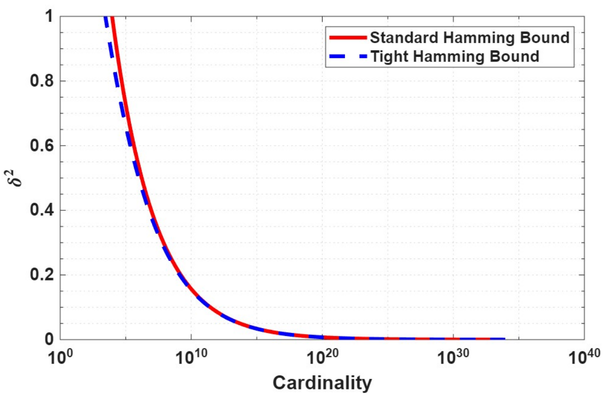
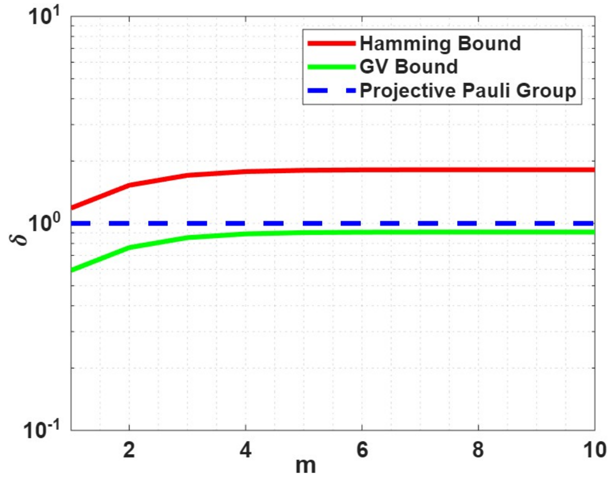


Fig. 3: Hamming bound (28) compared with tight Hamming bound (2) in $\mathcal{PU}_4$.

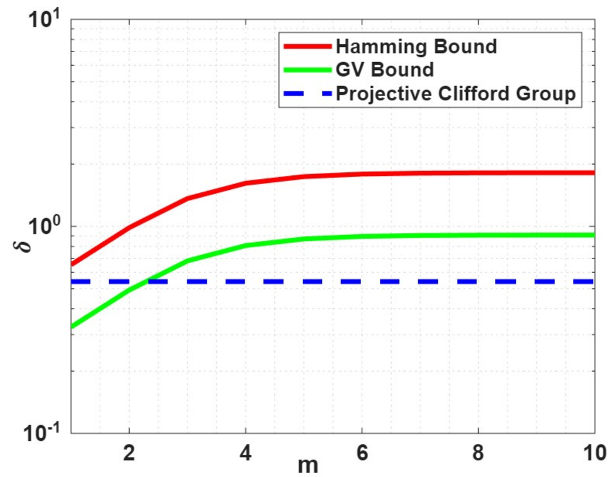
The upper and lower bounds on the kissing radius in $\mathcal{PU}_n$ provide geometric insight into the local packing density of codebooks. By comparing these bounds, we can analyze how tightly the unitary space is covered without overlap, thereby assessing the efficiency and optimality of the constructed projective codebooks. Fig. 2, for $n = 4$ and 500000 unitary matrices, illustrates the kissing radius bounds provided in Theorem 2. The bounds are compared to simulated midpoints between two randomly generated codewords. It is also compared with the estimate $\frac{\delta}{2}$, corresponding to the classical packing radius in flat geometry.

In Fig. 3, for $n = 4$ we compare the Hamming bound and the tight Hamming bound by kissing radius in terms of rate of codebook and square of minimum distance. We find that the kissing radius analysis is relevant only for small codebooks.

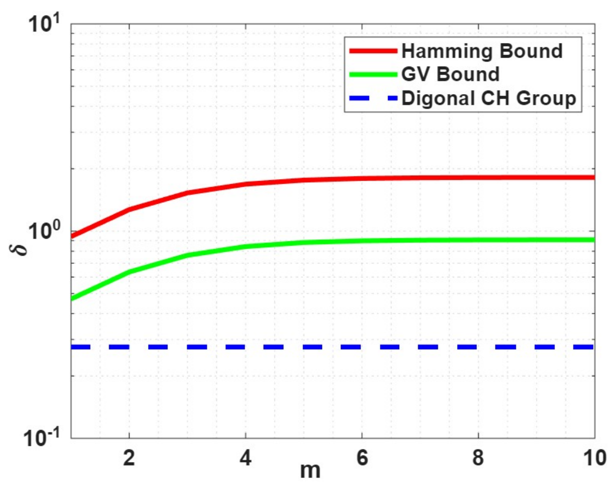
In Fig. 4, we compare minimum distances of codebooks in $\mathcal{PU}_n$ with the Hamming (28) and GV (27) bounds for the corresponding cardinality. The Hamming bound of provides a strict upper on minimum distance. The GV bound, in turn shows that there exists at least one codebook in $\mathcal{PU}_n$ whose



(a) Minimum distance of $\widetilde{\mathcal{P}}_n$, and GV (27) and Hamming (28) bounds in $n = 2^m$ dimensions.



(b) Minimum distance of $\tilde{\mathcal{G}}_n$, and GV (27) and Hamming (28) bounds in $n = 2^m$ dimensions.



(c) Minimum distance of $\widetilde{\mathcal{D}}_{n,3}$, and GV (27) and Hamming (28) bounds in $n = 2^m$ dimensions.

Fig. 4: Comparison of minimum distances and theoretical bounds for different code families.

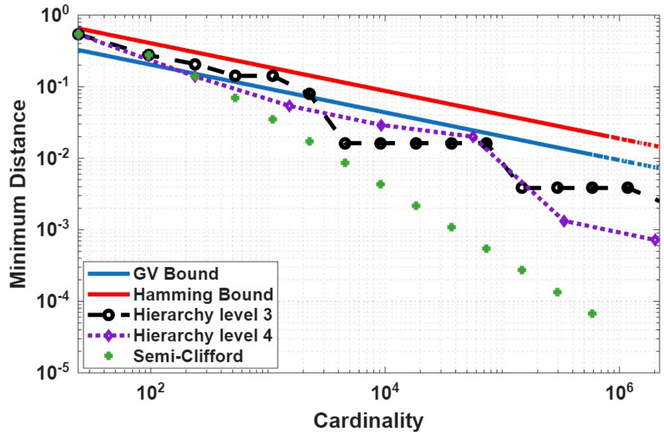


Fig. 5: Minimum distances of $\tilde{\mathcal{C}}_{2,3}^{l}$, $\tilde{\mathcal{C}}_{2,4}^{l}$ and $\tilde{\mathcal{C}}_{2,k}$ and comparison with GV (27) and Hamming (28) bounds in $\mathcal{PU}_2$.

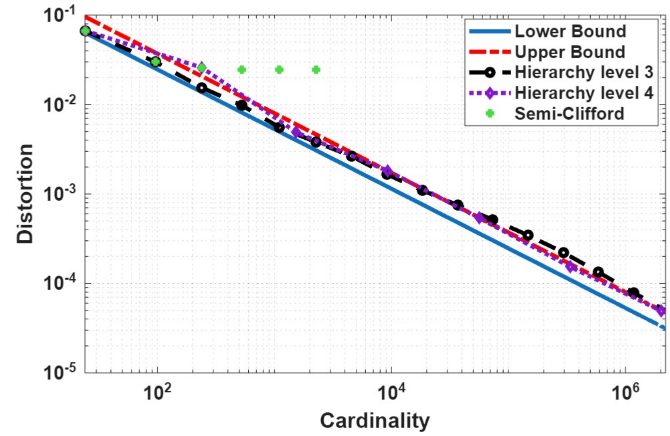


Fig. 6: Comparison of distortion of codebooks $\tilde{\mathcal{C}}_{2,3}^{l}$, $\tilde{\mathcal{C}}_{2,4}^{l}$, and codebooks $\tilde{\mathcal{C}}_{2,k}$ with the corresponding bounds (27) and (28) in $\mathcal{PU}_2$.

minimum distance is larger than the GV. As shown Fig. 4a, the minimum distance of Pauli matrices $\widetilde{\mathcal{P}}_n$ lie between these two bounds. The Pauli matrices are optimal due to their otrhoplectic structure. They outperform the general guarantee of the GV bound. In comparison, the Clifford groups of Fig. 4b, and the diagonal Clifford hierarcy of Fig. 4c are farther from optimum. For $m = 1$ and $m = 2$, the Clifford groups outperform the GV-bound. The diagonal Clifford hierarchy is systematically worse than the bound. Packing all the codewords in diagonal matrices compromizes the minimum distance.

In Fig. 5, minimum distances of codebooks of pructs of higher order Clifford hierarchy elements are shown. $\tilde{\mathcal{C}}_{2,3}^{l}$ codes for $l = 0$ to 15 stages of $\mathbf{T}$-gates and $\tilde{\mathcal{C}}_{2,4}^{l}$ codes for $l = 0$ to 6 stages of $\mathbf{S}$-gates in $\mathcal{PU}_2$ are considered. The results are compared to the GV and Hamming bounds. The minimum distances are obtained numerically by generating the codebooks and calculated their minimum distances. For small l, $\tilde{\mathcal{C}}_{2,3}^{l}$ outperform the GV bound, while $\tilde{\mathcal{C}}_{2,4}^{l}$ is slightly worse than this bound. For larger l, the minimum distances generically follow almost a similar slope as the bounds, with certain values of l being considerably better than some other.

Fig. 6 compares the distortion of the $\tilde{\mathcal{C}}_{2,3}^{l}$, $\tilde{\mathcal{C}}_{2,4}^{l}$, and semi-Clifford codebooks with the lower and upper bounds of the minimum distortion (39) in $\mathcal{PU}_2$. For the semi-Clifford codebook, we considered diagonal parts from levels $k = 2, ..., 7$. Here, we considered the quantization distortion of 500000 random unitary matrices for getting the results. As the figure illustrates, the semi-Clifford codebook exhibits a flooring behavior. After $k = 4$, the average distortion is only slightly improved when k increases. The reason is in the structure of the semi-Clifford hierarchy—with increasing k, there is a increasingly fine codebook of diagonal matrices. The off-diagonal directions, however, are simply given by the Clifford group, they are not getting richer with increasing l. In contrast the codebooks $\tilde{\mathcal{C}}_{2,3}^{l}$ and $\tilde{\mathcal{C}}_{2,4}^{l}$ consisting of products of higher level clifford hierarchy elements, exhibit performance very close to the bounds. The upper bound on the optimal distortion is given by the average distortion of random codebooks. Thus we observe that for small l, the $\tilde{\mathcal{C}}_{2,3}^{l}$ and $\tilde{\mathcal{C}}_{2,4}^{l}$ are better than a typical random codebook, and for large l, they are as good as a typical random codebook.

As the distortion performance of $\tilde{\mathcal{C}}_{2,3}^{l}$ and $\tilde{\mathcal{C}}_{2,4}^{l}$ is roughly equal, given the cardinality, the choice of using one rather than the other should be governed by implementation issues.

The covering radius quantifies the worst-case approximation error between any unitary in $\mathcal{PU}_2$ and its nearest codebook element, thus indicating how uniformly the codebook covers the unitary group space. In Fig. 7 we compare the covering radius of codebooks $\tilde{\mathcal{C}}_{2,3}^{l}$ for $l = 0$ to 15 stages of $\mathbf{T}$-gates and $\tilde{\mathcal{C}}_{2,4}^{l}$ for $l = 0$ to 6 stages of $\mathbf{S}$-gates with the theoretical lower bound (42) and the approximated covering radius (44). To find the covering radius of the of these codebooks, we generate 500000 unitary matrices then find the covering radius using (41). A smaller value of $\rho(\mathcal{C})$ indicates denser coverage and a more uniform sampling of $\mathcal{PU}_2$. For small l, the covering radius is close to the lower bound. It is interesting to note that for all considered values of l, the covering radius of these systematic codebooks are better than the approximate value (44), which is valid for random codebooks.

VII. Conclusion

In this paper, we considered quantum computation as a coding theoretical problem on the space $\mathcal{PU}_n$ of $n \times n$-dimensional projective unitary matrices. We first calculated the volume of $\mathcal{PU}_n$. Using this volume, we found the measure of small balls in $\mathcal{PU}_n$ with respect to the global phase invariant metric, and established the GV lower and Hamming upper bounds for codebooks in $\mathcal{PU}_n$. In addition, we provided the upper and lower bounds for the kissing radius of codes in $\mathcal{PU}_n$, which quantifies the maximum radius of non-overlapping metric balls. Based on normalized volumes of metric balls around

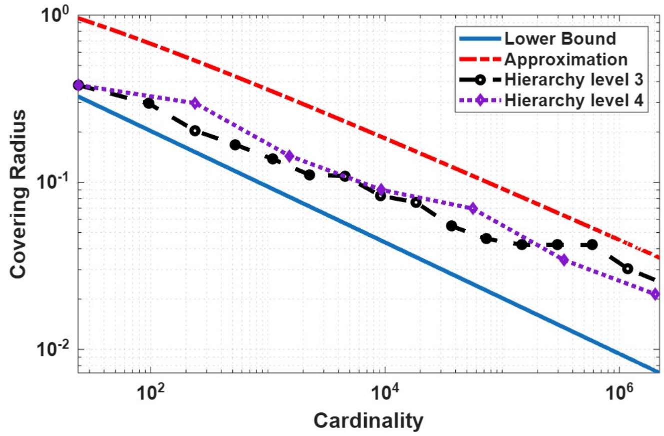


Fig. 7: Comparison of the covering radius between codebooks $\tilde{\mathcal{C}}^{l}_{2,3}$ and $\tilde{\mathcal{C}}^{l}_{2,4}$, along with the lower bound (42) and approximated of the covering radius (44) in $\mathcal{PU}_2$.

the kissing radius, we established bounds on the density of codes in $\mathcal{PU}_n$. Using the bound on code density, we provided an improved Hamming bound. Furthermore, we derived lower and upper bounds of the distortion-rate function over $\mathcal{PU}_n$, and provided a lower bound and an approximation for the covering radius. As examples of codebooks in $\mathcal{PU}_n$, relevant for quantum computation, We considered the projective Pauli group, the projective Clifford group, and the projective diagonal part of the Clifford hierarchy group, and found their minimum distances. As examples of larger cardinality codebooks, we considered single-qubit computational codebooks. In addition to higher Clifford-hierarchy level semi-Clifford circuits, we considered codebooks consisting of products of a finite number of 3rd level and 4th level Clifford hierarchy elements. The comparison of numerical performance results to bounds show that increasing the hierarchy level alone does not improve performance much. In contrast, products of multiple higher hierarchy level elements leads to performance which is comparable, and even slightly better than that of random codebooks.

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