Bounds in the Projective Unitary Group with Respect to Global Phase Invariant Distance

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***Abstract*—We consider a global phase-invariant distance in the projective unitary group $\mathcal{PU}_n$, relevant for universal quantum computing. We obtain the volume and measure of small metric ball in $\mathcal{PU}_n$ and derive the Gilbert-Varshamov and Hamming bounds in $\mathcal{PU}_n$. In addition, we provide upper and lower bounds for the kissing radius of the codebooks in $\mathcal{PU}_n$ as a function of the minimum distance. Using the lower bound of the kissing radius, we find a tight Hamming bound. Also, we establish bounds on the distortion-rate function for quantizing a source uniformly distributed over $\mathcal{PU}_n$. As example codebooks in $\mathcal{PU}_n$, we consider the projective Pauli and Clifford groups, as well as the projective group of diagonal gates in the Clifford hierarchy, and find their minimum distances. Finally, we verify the analytical results by simulation.**

***Index Terms*—Projective Unitary Group, Volume, Kissing Radius, Hamming bound, and Gilbert-Varshamov bound.**

I. INTRODUCTION

In quantum computing, the design of quantum algorithms can be seen as a decomposition of a unitary matrix using a set of universal gates. It is well known that the set of Clifford gates combined with a non-Clifford gate forms a set of universal gates for quantum computation [1]. Exact decomposition or approximation of an arbitrary unitary matrix using a set of universal gates has been addressed in [2]–[4]. In [2], the total number of single-qubit gates that can be represented by the Clifford+T gates is calculated. An algorithm for finding a T-optimal approximation of single-qubit Z-rotations using Clifford+T gates is proposed in [3], which is capable of handling errors down to 10^{-15}. Approximating an arbitrary single-qubit gate from the special unitary group using Clifford+T gates, up to any given error threshold, is proposed in [4].

In quantum computation, the overall phase is irrelevant since it does not affect the measurable properties of a quantum system [1]. Hence, the gate approximation should be considered in the projective unitary group $\mathcal{PU}_n$ rather than in the unitary group or the special unitary group. $\mathcal{PU}_n$ consists of the equivalence classes of $n \times n$ unitary operations that differ by a global phase [5]. This makes the projective unitary group fundamental for constructing reliable quantum gates and enabling universal quantum computation.

A global phase invariant metric, which is suitable for $\mathcal{PU}_n$, is considered in [3], [6]–[8]. In [6], this metric is used for constructing the optimal fault-tolerant approximation of arbitrary gates with a set of discrete universal gates. Using this metric, the error approximations of universal gates are discussed in [7]. Furthermore, the T-count and T-depth of any multi-qubit unitary, which are crucial for optimizing quantum circuits, are analyzed in [8].

The volume of a small ball is needed for deriving the bounds on packing and covering problems. The volume of a small ball in the unitary group, Grassmannian, and Stiefel manifolds are well understood [9]–[13]. However, $\mathcal{PU}_n$ remains largely unexplored in the literature, particularly in terms of volume analysis and theoretical bounds.

Motivated by this, we consider the global phase-invariant distance in $\mathcal{PU}_n$ and compute the volume of a small ball. Using this volume, we derive the Hamming upper and Gilbert-Varshamov (GV) lower bounds. In addition, we obtain upper and lower bounds for the kissing radius as a function of the minimum distance of the codebook in $\mathcal{PU}_n$, and establish a tight Hamming bound. We derive upper and lower bounds for the distortion rate function. Furthermore, as examples of codebooks in $\mathcal{PU}_n$, we consider the projective Pauli group, the projective Clifford group, and the group of projective diagonal gates in the Clifford hierarchy, and determine their minimum distances. Finally, through the numerical results, we show the validity of our analyses.

The rest of this paper is organized as follows: Section II provides preliminaries. We derive the volume of metric balls for $\mathcal{PU}_n$ in Section III, and give the Hamming upper and GV lower bounds. Section IV provides the upper and lower bounds for the kissing radius as a function of the minimum distance. Also, we obtain bounds on the distortion-rate function in $\mathcal{PU}_n$. Section V discusses the simulation results, and Section VI concludes the paper.

II. PRELIMINARIES

A. The Projective Unitary Group

The projective unitary group $\mathcal{PU}_n$ is a group of $n \times n$ complex valued matrices which can be represented in the quotient geometry as $\mathcal{U}_n/\mathcal{U}_1$, where $\mathcal{U}_n$ denotes the unitary group. The dimension of $\mathcal{PU}_n$ is $n^2 - 1$, and the elements are equivalence classes:

$$\mathcal{PU}_n = \{\alpha \mathbf{U} \;\mid\; \mathbf{U} \in \mathcal{U}_n \;\text{ and }\; |\alpha| = 1\}, \tag{1}$$

which can be represented by any unitary matrix $\mathbf{U}$ belonging to the class.

In this paper, we use the following metric [6]:

$$d(\mathbf{U},\mathbf{V}) = \sqrt{1-\frac{1}{n}|\mathrm{Tr}\,(\mathbf{U}^H\mathbf{V})|}, \tag{2}$$

for $\mathbf{U},\mathbf{V} \in \mathcal{PU}_n$, where $(.)^H$ denotes the Hermitian conjugate. This is a metric on $\mathcal{PU}_n$, as it does not depend on the overall phase of the representation $\mathbf{U}$ of an element in $\mathcal{PU}_n$.

In [1], the relationship between the operator norm $d_O(\mathbf{U},\mathbf{V}) = \max_\psi \|(\mathbf{U}-\mathbf{V})|\psi\rangle\|$, where the maximum is over all pure states $|\psi\rangle$, and the trace distance $\mathrm{Tr}\left(\sqrt{(\mathbf{U}-\mathbf{V})^H(\mathbf{U}-\mathbf{V})}\right)$ for single-qubit rotations is discussed, particularly in the context of approximating unitary operators. In determining these distances, the global phase of a unitary matrix plays a significant role. For example, for both of these metrics, the distance between $\mathbf{U}$ and $-\mathbf{U}$ is maximal, while for (2), their distance is zero, as they come from the same equivalence class. The phase invariant metric provides a notable advantage in finding optimal approximations, as it is invariant under global phase shifts.

B. Packing and Covering Problems

The packing problem is to fit a maximal set of non-overlapping balls of a given radius into the space. A finite subset of points in manifold $\mathcal{M}$

$$\mathcal{C} = \{\mathbf{C}_1, \ldots, \mathbf{C}_{|\mathcal{C}|}\} \subset \mathcal{M}, \tag{3}$$

is a $(|\mathcal{C}|, \delta)$-code, with

$$\delta = \min\{d(\mathbf{C}_i,\mathbf{C}_j) : \mathbf{C}_i,\mathbf{C}_j \in \mathcal{C}, i \neq j\} \tag{4}$$

the minimum distance.

The standard Hamming bound is a packing bound that provides an upper bound for the cardinality of a code given its minimum distance [13].

In non-Euclidean geometry, the maximum radius of the non-overlapping balls, known as the kissing radius, may be larger than half of the minimum distance. The kissing radius of a code $\mathcal{C}$ is defined as

$$\varrho = \sup_{\substack{B_{\mathbf{C}_l}(R)\cap B_{\mathbf{C}_k}(R)=\emptyset \\ \forall (k,l), k\neq l}} R, \tag{5}$$

where

$$B_{\mathbf{C}_i}(R) = \{\mathbf{P} \in \mathcal{M} \,:\, d(\mathbf{P},\mathbf{C}_i) \leq R\} \tag{6}$$

is the metric ball of radius r centered on the codeword $\mathbf{C}_i$.

The covering problem is to find the minimum number of overlapping balls of a given radius, required to cover the entire space. The GV bound is a covering bound that provides a lower bound on the cardinality of the code, given its minimum distance [14].

C. Codebooks in the Projective Unitary Group

In this section, we consider three different codebooks in $\mathcal{PU}_n$: the projective Pauli group, the projective Clifford group, and the diagonal part of the Clifford Hierarchy. These codebooks play a crucial role in quantum theory [15]–[18]. In the following, we consider unitary matrices of dimension $n = 2^m$, with $m = 1, 2, \ldots$.

1) Projective Pauli Group: The 2×2 Pauli matrices are:

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The n-dimensional Pauli group is then defined as the set

$$\mathcal{P}_n = \{\pm\mathbf{D}(\mathbf{a},\mathbf{b}), \pm i\mathbf{D}(\mathbf{a},\mathbf{b})\},$$

where

$$\mathbf{D}(\mathbf{a},\mathbf{b}) = \mathbf{X}^{a_1}\mathbf{Z}^{b_1} \otimes \mathbf{X}^{a_2}\mathbf{Z}^{b_2} \otimes \cdots \otimes \mathbf{X}^{a_m}\mathbf{Z}^{b_m},$$

with binary vectors $\mathbf{a} = [a_1, ..., a_m]^T, \mathbf{b} = [b_1, ..., b_m]^T$.

The projective Pauli group is defined as $\widetilde{\mathcal{P}}_n = \mathcal{P}_n/\mathbb{Z}_4$, where the center is

$$\mathbb{Z}_{2^{k+1}} \triangleq \{e^{\frac{2\pi i}{2^k}q}\mathbf{I}_n \mid q = 0, 1, ..., 2^k - 1\}, \tag{7}$$

with $\mathbf{I}_n$ the identity matrix. Note that $\widetilde{\mathcal{P}}_n$ has cardinality $n^2 = 2^{2m}$.

2) Projective Clifford Group: The second level of the Clifford hierarchy is the Clifford group which is defined as

$$\mathcal{G}_n = \{\mathbf{G} \in \mathcal{U}_n \mid \mathbf{G}^H\mathcal{P}_n\mathbf{G} \subset \mathcal{P}_n\}. \tag{8}$$

The center of this group is given by $\mathcal{U}_1$. Accordingly, the projective Clifford group is defined as $\tilde{\mathcal{G}}_n = \mathcal{G}_n/\mathcal{U}_1$. The cardinality of $\tilde{\mathcal{G}}_n$ is given by [19]

$$\left|\tilde{\mathcal{G}}_n\right| = 2^{m^2+2m}\prod_{i=1}^{m}(2^{2i}-1). \tag{9}$$

3) Projective Diagonal Part of the Clifford Hierarchy: The diagonal Clifford hierarchy of level k denoted by $\mathcal{D}_k$ forms a group and can be generated by the rotations of $\mathbf{Z}_j[\frac{\pi}{2^k}] = \exp\left(\frac{i\pi}{2^k}\mathbf{Z}_j\right)$, where $\mathbf{Z}_j$ is the Pauli $\mathbf{Z}$ acting on the jth qubit [20]. In particular, for m-qubit quantum system with $m \geq k$, the following gates construct $\mathcal{D}_k$

$$\left\langle \mathbf{Z}_i\left[\frac{\pi}{2^k}\right], \mathbf{\Lambda}^1_{i_1,i_2}\left(\mathbf{Z}\left[\frac{\pi}{2^{k-1}}\right]\right), \ldots, \mathbf{\Lambda}^{k-1}_{i_1,\ldots,i_k}\left(\mathbf{Z}[\frac{\pi}{2}]\right)\right\rangle,$$

where $\mathbf{\Lambda}^k(\mathbf{U})$ denotes the k-controlled $\mathbf{U}$ gate, acting on $k+1$ qubits. Here, $i, i_1, i_2, \ldots, i_k$ run over all qubits. For $m < k$ a similar set of generating gates truncated at $m' = m-1$ control gates construct $\mathcal{D}_k$. The projective diagonal part of the Clifford hierarchy is defined as $\tilde{\mathcal{D}}_k = \mathcal{D}_k/\mathbb{Z}_{2^k}$. The cardinality of $\tilde{\mathcal{D}}_k$ is given by [20]

$$|\tilde{\mathcal{D}}_k| = \prod_{j=0}^{\min(k-1,m-1)} \left(2^{k-j}\right)^{\binom{m}{j+1}}. \tag{10}$$

III. VOLUME OF THE PROJECTIVE UNITARY GROUP

Using this volume, we derive a measure of small metric ball in $\mathcal{PU}_n$ with respect to the metric given by (2). In addition, we provide the Hamming upper and GV lower bounds in $\mathcal{PU}_n$.

The Euclidean $(D-1)$-sphere of radius R in $\mathbb{R}^D$ is defined as $\mathcal{S}^{D-1}(R) = \left\{\mathbf{x} \in \mathbb{R}^D \,\middle|\, \|\mathbf{x}\|_2 = R\right\}$. The volume of $\mathcal{S}^{D-1}(R)$ is given by

$$\mathrm{V}_D(R) = \frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}+1\right)}R^D. \tag{11}$$

Also, the volume of the unitary group $\mathcal{U}_n$ is [13]

$$\mathrm{Vol}(\mathcal{U}_n) = \frac{(2\pi)^{\frac{n(n+1)}{2}}}{\prod_{i=1}^{n}(i-1)!}. \tag{12}$$

Theorem 1. *The volume of the $\mathcal{PU}_n$ is*

$$\mathrm{Vol}\left(\mathcal{PU}_n\right) = \frac{(2\pi)^{\frac{n(n+1)}{2}}}{2\pi\sqrt{n}\prod_{i=1}^{n}(i-1)!}. \tag{13}$$

Proof: According to the [5], the volume of a homogeneous quotient space $\mathcal{G}/\mathcal{K}$ arising from the free and proper action of subgroup $\mathcal{K}$ on group $\mathcal{G}$ is $\mathrm{Vol}\left(\mathcal{G}\right)/\mathrm{Vol}\left(\mathcal{K}\right)$. The volume of $\mathrm{Vol}(\mathcal{U}_n)$ is given by (12). The subgroup forming the cosets in (1) is isomorphic to $\mathcal{U}_1$, but strictly speaking not isometric. To find a metric on this subgroup, consider $\mathbf{X} = e^{i\theta}\mathbf{I}_n$ and $\mathbf{X}' = \mathbf{X} + d\mathbf{X}$ where $d\mathbf{X} = ie^{i\theta}\mathbf{I}_n\, d\theta$. The infinitesimal distance is given by

$$(ds)^2_{\mathcal{U}_1} = \left\|\mathbf{X} - \mathbf{X}'\right\|_F^2 = \left\|d\mathbf{X}\right\|_F^2 = n\, d^2\theta. \tag{14}$$

Therefore, the subgroup divided away is isometric to a circle with radius $\sqrt{n}$, and the volume of subgroup $\mathcal{K}$ in $\mathcal{U}_n$ is $\mathrm{Vol}(\mathcal{K}) = \int_0^{2\pi}\sqrt{n}d\theta = 2\pi\sqrt{n}$. The statement follows directly. ■

The measure of the metric ball in the manifold $\mathcal{M}$, considering the Frobenius norm, is defined as

$$\mu_{\mathrm{F}}\left(B(R)\right) = \frac{\mathrm{Vol}\left(B(R)\right)}{\mathrm{Vol}(\mathcal{M})}, \tag{15}$$

where $\mathrm{Vol}\left(B(R)\right)$ is the volume of the ball with radius R in the manifold. For the measure of the metric ball in $\mathcal{PU}_n$ we have

Corollary 1. *As $R \to 0$, the measure of a metric ball $B(R)$ in $\mathcal{PU}_n$ with respect to the global phase invariant distance metric* (2) *is*

$$\mu_{\mathrm{d}}(B(R)) = c_n\, R^D(1 + \mathcal{O}(R^2)) \tag{16}$$

where $c_n = \frac{(2\pi)^{-\frac{(n-1)}{2}} n^{\frac{n^2}{2}}}{\Gamma(\frac{n^2-1}{2}+1)}\prod_{i=1}^{n}(i-1)!$, and $D = n^2 - 1$ is the dimension of $\mathcal{PU}_n$.

Proof:

The measure of metric ball in $\mathcal{PU}_n$ with respect to the metric (2) can be written as

$$\begin{aligned} F_d\left(R\right) &= \Pr\{d \leq R\} = \Pr\left\{\frac{d_F}{\sqrt{2n}} \leq R\right\} \\ &= \mu_{\mathrm{F}}\left(B\left(\sqrt{2n}R\right)\right), \end{aligned} \tag{17}$$

where d_{F} denotes the Frobenius distance. The volume of a small ball can be well approximated by the volume of a ball of equal radius in the tangent space [13] as

$$\mathrm{Vol}\left(B(R)\right) = \mathrm{V}_D(R)\left(1 + O(R^2)\right). \tag{18}$$

Substituting (18) and (13) in (15) and considering (17) completes the proof.

■

The GV and Hamming bounds provide lower and upper bounds on the cardinality of a codebook in the manifold [9]. In the following, we provide these bounds for $\mathcal{PU}_n$.

Corollary 2. *There exists a codebook $\mathcal{C}$ in $\mathcal{PU}_n$ with cardinality $|\mathcal{C}|$ and the minimum distance δ with respect to the metric distance* (2) *such that*

$$\frac{1}{\mu_d(B(\delta))} \leq |\mathcal{C}|. \tag{19}$$

Also, for any $\left(|\mathcal{C}|, \delta\right)$-codebook in $\mathcal{PU}_n$

$$|\mathcal{C}| \leq \frac{1}{\mu_d(B(\frac{\delta}{2}))}. \tag{20}$$

IV. KISSING RADIUS AND DISTORTION RATE OF THE $\mathcal{PU}_n$

A. Kissing Radius Bounds of Projective Unitary Group

In this section, we derive upper and lower bounds for the kissing radius ϱ as a function of the minimum distance of a code in $\mathcal{PU}_n$ with respect to the global phase-invariant distance.

Moreover, we establish a tight Hamming bound in this context, using the density $\Delta(\mathcal{C})$ of a code. For a code with cardinality K and kissing radius ϱ, the density is [13]:

$$\Delta(\mathcal{C}) = K\mu_d(B(\varrho)).$$

Lemma 1. *Let $\mathbf{U}, \mathbf{V} \in \mathcal{PU}_n$, and define $\mathbf{W} = \mathbf{U}^H\mathbf{V}$. Considering the metric* (2)*, the geodesics midpoint between $\mathbf{U}$ and $\mathbf{V}$ is given by*

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Omega}\sqrt{\mathbf{L}}\boldsymbol{\Omega}^H, \tag{21}$$

where $\mathbf{W} = \boldsymbol{\Omega}\mathbf{L}\boldsymbol{\Omega}^H$ with $\mathbf{L} = \mathrm{diag}\left(e^{j\theta_1}, ..., e^{j\theta_n}\right)$.

Proof: Since $\mathcal{PU}_n$ is a Lie group, its geodesics can be described using its Lie algebra $\mathfrak{pu}(n)$. As discussed in [21], the Lie algebra of $\mathcal{PU}_n$ is $\mathfrak{pu}(n) \cong \mathfrak{u}(n)/\{ia\mathbf{I}\}$, where $\mathfrak{u}(n)$ is the Lie algebra of $\mathcal{U}_n$. The geodesic curve in $\mathcal{PU}_n$ is given by

$$\boldsymbol{\gamma}(t) = \mathbf{U}e^{t(\mathbf{A}+ia\mathbf{I})} = \mathbf{U}e^{t\mathbf{A}'}, \qquad 0 \leq t \leq 1$$

where $\mathbf{A}, \mathbf{A}' \in \mathfrak{u}(n)$ is a skew-Hermitian matrix such that $\boldsymbol{\gamma}(0) = \mathbf{U}, \boldsymbol{\gamma}(1) = \mathbf{V} = \mathbf{U}e^{\mathbf{A}'}$. The geodesics midpoint is given by $\boldsymbol{\gamma}(1/2)$ which can be written in the form of $\mathbf{M}$ given in (21). We skip the details due to space limitations. ■

The kissing radius of a given code is hard to determine since it depends on the minimum distance of the code and principal angles between codewords [22]. The following theorem provides lower and upper bounds for the kissing radius of a code in $\mathcal{PU}_n$ with the help of Lemma 1.

Theorem 2. *For any code $\left(|\mathcal{C}|, \delta\right) \in \mathcal{PU}_n$, the kissing radius ϱ is bounded as*

$$\underline{\varrho} \leq \varrho \leq \bar{\varrho},$$

where $\underline{\varrho} = \sqrt{1 - \sqrt{1 - \frac{\delta^2}{2}}}$ and $\bar{\varrho} = \sqrt{1 - \sqrt{\frac{1+(1-\delta^2)^2}{2}}}$. The corresponding bounds on codebook density are

$$|\mathcal{C}|\,\mu_d(B(\underline{\varrho})) \leq \Delta(\mathcal{C}) \leq \min\{1, |\mathcal{C}|\,\mu_d(B(\bar{\varrho}))\},. \tag{22}$$

Proof: The minimum distance and kissing radius can be written in terms of $\theta_i, i = 1, ..., n$ in Lemma 1. We then consider optimizing the kissing radius given the minimum distance. The details are omitted due to space constraints. ■

One of the central problems in coding theory is determining the maximum size of a codebook for a given minimum distance. Using the normalized volume of the metric ball $\mu_d(B(r))$, as given in Corollary 1, and applying Theorem 2, we obtain the following refined version of the Hamming bound:

Corollary 3. *For any* $(|\mathcal{C}|, \delta)$*-code in* $\mathcal{PU}_n$*, we have*

$$|\mathcal{C}| \leq \frac{1}{\mu_d(B(\underline{\varrho}))}, \tag{23}$$

where $\underline{\varrho}$ *is given in Theorem 2.*

Proof: From (22), $|\mathcal{C}| \, \mu_d(B(\underline{\varrho})) \leq \Delta(\mathcal{C}) \leq 1$, and we have $\frac{\delta}{2} \leq \underline{\varrho}$. This implies that $|\mathcal{C}| \leq \frac{1}{\mu_d(B(\underline{\varrho}))} \leq \frac{1}{\mu_d(B(\frac{\delta}{2}))}$. ■

B. Distortion-Rate function in $\mathcal{PU}_n$

In universal quantum computation, the goal is to approximate a given unitary gate by the closest element of a universal gate set. This quantization problem, approximating the target using available gates, is directly related to rate-distortion theory. The distortion–rate function is defined as [12]

$$\mathcal{D}^*(K) = \inf_{\mathcal{C}:|\mathcal{C}|=K} \mathcal{D}(\mathcal{C}), \tag{24}$$

where

$$\mathcal{D}(\mathcal{C}) = \mathbb{E}\left[\min_{\mathbf{P}\in\mathcal{C}} d^2(\mathbf{P}, \mathbf{Q})\right], \tag{25}$$

where $\mathcal{C}$ in $\mathcal{PU}_n$ with cardinality $|\mathcal{C}| = K$. Here, $\mathbf{Q}$ is an arbitrary point in the space.

Based on the volume of $\mathcal{PU}_n$ given in Corollary 1, the distortion–rate tradeoff is characterized by establishing lower and upper bounds on the distortion–rate function. For a codebook $\mathcal{C}$ with sufficiently large cardinality K, the distortion–rate function over the $\mathcal{PU}_n$, with global phase invariant distance, can be bounded as

$$\frac{D}{D+2}\left(c_n K\right)^{-\frac{2}{D}} \leq \mathcal{D}^*(K) \leq \frac{2\Gamma\left(\frac{2}{D}\right)}{D}\left(c_n K\right)^{-\frac{2}{D}}\left(1 + o(1)\right), \tag{26}$$

where $D = n^2 - 1$ and c_n given in Corollary 1. This is an extension of the results in [12] for the Grassmannian manifold to $\mathcal{PU}_n$.

As discussed in [12], for a code (K, δ) in $\mathcal{PU}_n$ the distortion is upper bounded as

$$\mathcal{D}(\mathcal{C}) \leq \left(\frac{\delta^2}{4} - 1\right) K \mu_d(B(\delta/2)) + 1\,. \tag{27}$$

Note that in a flat space, the packing radius is $\frac{\delta}{2}$. However, in a non-flat geometry $\frac{\delta}{2} \leq \varrho \leq \delta$. Therefore, using the lower bound of the kissing radius, we can have a tighter upper bound on the distortion than (27).

C. Codebooks of Projective Unitary Matrices

The minimum distance of quantum codebooks plays a pivotal role in facilitating both error correction and error detection [23]–[25]. Here, we find the minimum distance of the example codebooks in $\mathcal{PU}_n$.

Proposition 1. *The minimum distances of* $\widetilde{\mathcal{P}}_n$ *and* $\tilde{\mathcal{G}}_n$ *considering the global phase invariant distance are*

$$\delta_p = 1\,, \qquad \delta_c = \sqrt{1 - \frac{1}{\sqrt{2}}}\,, \tag{28}$$

respectively.

Proof: As $\widetilde{\mathcal{P}}_n$ is a group under multiplication, let $\mathbf{U}^H\mathbf{V} = \mathbf{W} = e^{\frac{2\pi i}{2^k}q}\mathbf{D}(\mathbf{a}, \mathbf{b}) \in \widetilde{\mathcal{P}}_n$, i.e., $d(\mathbf{U}, \mathbf{V}) = d(\mathbf{I}, \mathbf{W})$. Then $|\mathrm{Tr}\left(\mathbf{I}^H\mathbf{W}\right)| = |\mathrm{e}^{\frac{2\pi i}{2^k}q}\,\mathrm{Tr}\left(\mathbf{D}(\mathbf{a}, \mathbf{b})|$. We have

$$|\mathrm{e}^{\frac{2\pi i}{2^k}q}\,\mathrm{Tr}\left(\mathbf{D}(\mathbf{a}, \mathbf{b})\right| = \begin{cases} n, & \mathbf{a} = \mathbf{b} \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

From (2) and (29), the distance of any two codewords in $\widetilde{\mathcal{P}}_n$ is zero or 1. Hence, $\delta_p = 1$. We skip the proof for the Clifford group because of space limitations. ■

Proposition 2. *The minimum distance of* $\tilde{\mathcal{D}}_k$ *with respect to* (2) *is*

$$\delta_d = \sqrt{1 - \cos\left(\frac{\psi_k}{2}\right)}, \text{ where } \psi_k = \frac{2\pi}{2^k}.$$

Proof: We skip the proof because of space limitations. ■

Note that using (27) and Propositions 1 and 2, we can obtain distortion upper bounds for codebooks $\widetilde{\mathcal{P}}_n$, $\tilde{\mathcal{G}}_n$ and $\tilde{\mathcal{D}}_k$.

V. SIMULATION RESULTS

In this section, we verify the correctness of our analyses in $\mathcal{PU}_n$ using numerical results. First, in Fig. 1, we consider the measure of the ball in $\mathcal{PU}_n$ given by Corollary 1. This figure illustrates the small ball volume in $\mathcal{PU}_n$ for $n = 2$ in terms of the global phase invariant distance. The simulation results are obtained by averaging over 10^8 unitary matrices generated uniformly at random with the Haar measure, following [26]. Note that due to the quotient structure, this also provides the Haar measure in $\mathcal{PU}_n$. The simulations results for small values of the distance matches with the theoretical evaluation.

To compare to the example codebooks of $\mathcal{PU}_n$, we use Corollary 2 to find the bounds on the minimum distance of the codes given the cardinality. Using (19) and (20), we can find lower and upper bounds on the minimum distance of the code with given cardinality:

$$\delta_{\mathrm{GV}} = (|\mathcal{C}|c_n)^{-\frac{1}{D}}$$
$$\delta_{\mathrm{H}} = 2(|\mathcal{C}|c_n)^{-\frac{1}{D}},$$

where $|\mathcal{C}|$ is the cardinality of the code. Fig. 2 compares the ratio of δ_{GV}, δ_{H}, and minimum distance of the projective Pauli, Clifford and the diagonal part of the Clifford hierarchy with

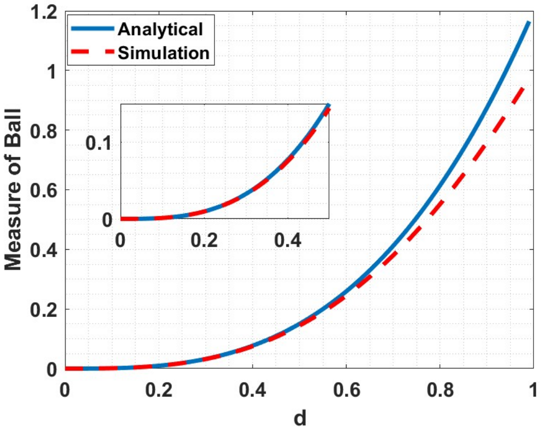

Fig. 1. Theoretical and simulation results comparison of the measure of the ball given by corollary 1.

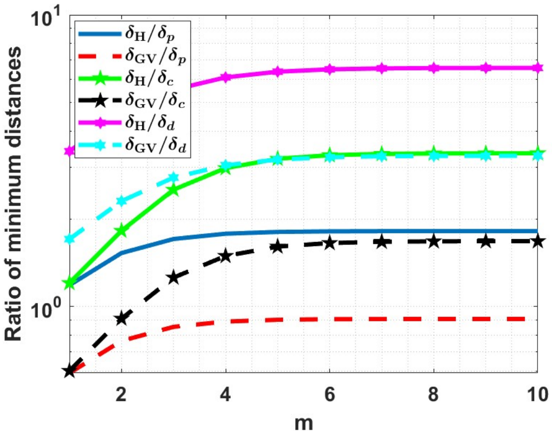

Fig. 2. Comparison of the ratio of minimum distances with given cardinality of the code resulted by Hamming and GV bounds, and the minimum distances of $\widetilde{\mathcal{P}}_n, \tilde{\mathcal{G}}_n$, and $\tilde{\mathcal{D}}_n$ with $k = 3$, $n = 2^m$.

$k = 3$. The diagonal part of the Clifford hierarchy performs worse than the other codebooks. For small values of n, the minimum distance of the projective Clifford group is slightly better than the GV bound, however, by increasing n it becomes less. However, the minimum distance of the $\widetilde{\mathcal{P}}_n$ is between the bounds.

Since the cardinality of the considered codebooks are very high, finding the actual quantization error by simulation is not trivial. In Fig. 3, we compare the distortion rate bounds given in (26) with the distortion rate of the random unitary matrices for $n = 2, 4$ and 8. To find the distortion rate of the random unitary matrices, we generate 500000 unitary matrices and select K of them and find the distortion rate. This procedure is done for $500000/K$ different sets of matrices and finally averaged out to give the distortion rate for this specific K. Here, we assume $K = 2^i$ for $i = 1, ..., 10$. Simulation results confirm that the bounds in (26) hold for sufficiently large values of K.

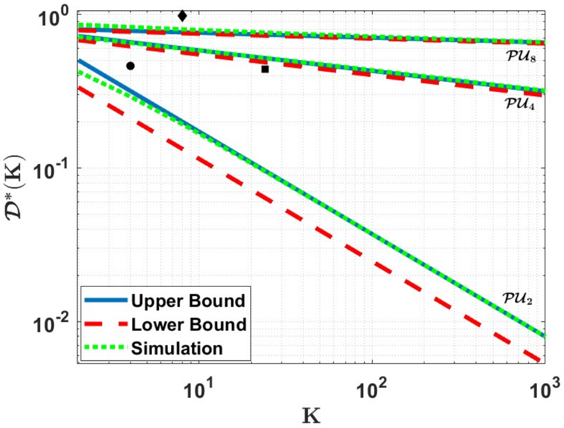

Fig. 3. Bounds on the distortion–rate function.

TABLE I
Tight Distortion upper bound given by a lower bound of the kissing radius for example codebooks.

n	$\widetilde{\mathcal{P}}_n$	$\tilde{G}_n$	$\tilde{\mathcal{D}}_3$
2	0.4618	0.4410	0.9749
4	0.9960	0.9998	1
8	1	1	1

Also, in Fig. 3, the upper bounds given by (27) for $\widetilde{\mathcal{P}}_2$, $\tilde{\mathcal{G}}_2$, and $\tilde{\mathcal{D}}_2$ with $k = 3$, are shown with circle, square, and diamond shapes, respectively. To find these points, we replaced $\delta/2$ with the lower bound of the kissing radius for each codebook. Table I summarizes the distortion bounds for these codebooks.

VI. Conclusion

In this paper, we calculated the volume of $\mathcal{PU}_n$. Using this volume, we found the measure of small balls in $\mathcal{PU}_n$ with respect to the global phase invariant metric, and established the GV lower and Hamming upper bounds for codebooks in $\mathcal{PU}_n$. In addition, we provided the upper and lower bounds for the kissing radius of codes in $\mathcal{PU}_n$, which quantifies the maximum radius of non-overlapping metric balls. Based on normalized volumes of metric balls around the kissing radius, we established bounds on the code density of codes in $\mathcal{PU}_n$. Using the bound on code density, we provided an improved Hamming bound. Furthermore, we derived lower and upper bounds of the distortion-rate function over $\mathcal{PU}_n$. We considered the projective Pauli group, the projective Clifford group, and the projective diagonal part of the Clifford hierarchy group as examples of codebooks in $\mathcal{PU}_n$ and found their minimum distances. Finally, we provided simulation results.

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