Quantum Error Correction for 2nd Generation Quantum Repeaters

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***Abstract*—We consider 2nd generation (2G) Quantum Repeaters (QR) for creating long-distance entanglement in quantum networks. Combining a distance-dependent depolarizing error model with the non-local Bell-state purification procedure required by 2G QRs leads to an error model consisting of correlated and biased errors. To correct correlated errors, non-symmetric CSS codes with joint decoding between stations can be used. The dominating errors are biased, such that different repeater stations suffer from different types of errors. To mitigate this, different quantum codes can be used at the stations, optimized for the specific error model of the station. To comply with the 2G QR procedure, the codes used in neighboring stations must allow for the transversal implementation of non-local logical CNOT gates across the two stations, or alternatively, non-local CZ gates combined with logical Hadamards. We provide a complete characterization of pairs of Calderbank-Steane-Shor (CSS) codes that allow CNOT- or CZ-transversality, and examine an explicit family of mirrored CSS codes allowing CZ-transversality. We verify Hadamard gate transversality using our framework and show the importance of the logical qubit mapping matrix. Also, we conclude that using different QECCs does not lead to universal computation with the Clifford + T-gate set. Finally, we study the entanglement generation rate (EGR) in 2G QRs with limited quantum memory, minimizing the number of intermediate stations for a given fidelity and EGR. By simulation we observe that non-symmetric and mirrored structure QECCs outperform the conventional approach of using symmetric CSS codes at the repeater stations.**

***Index Terms*—Quantum repeater, Quantum error correction, CSS codes, Bell state purification, Transversality, Key generation rate.**

I. INTRODUCTION

QUANTUM Repeaters (QRs) are essential for building long-range quantum communication networks, including the quantum internet. As in classical communication systems, repeaters can be used to increase the communication rate in long-distance quantum communication [1].

The difference between classical and quantum communication is the no-cloning theorem, which states that it is impossible to create an identical copy of an unknown quantum state [2].

QRs overcome this challenge, extending the range of quantum communications without violating the no-cloning theorem. This makes them crucial for building efficient and secure long-distance quantum networks [1], [3]–[5]. At the present time, three generations of QRs have been considered, based on three different communication protocols. The first-generation, known as *trusted node repeaters* [6], [7], use heralded entanglement purification and entanglement swapping to reduce erasure and operational errors [8], [9]. The main disadvantage of 1st generation QRs is the requirement of long-range entangled quantum states, i.e., Bell states with sufficiently low physical error rate.

Second generation (2G) QRs use QECC, in addition to entanglement purification and swapping, for creating trusted long-range entanglement. With QECC, we can tolerate more errors during transmission and increase the communication distance between stations while keeping the overall fidelity in reliable range [1], [10]. QECCs encode many physical qubits into logical qubits and protect logical information with physical redundancy [11]. However, 2nd generation QRs need classical communication between the stations, and the rate of end-to-end entanglement generation is limited due to the time-consuming Bell state purification process [12].

In third generation QRs, the message is encoded by QECC, and directly transmitted between stations [10]. In each station, the message is error corrected to recover from erasure errors and transmission errors. This generation depends mainly on QECCs, which require high-fidelity local quantum gates and large-scale QECCs.

In this paper, we concentrate on 2G QRs, whose requirements are likely to be achievable in near-term quantum devices [13]. First, we formulate an explicit error model for 2G QRs, taking into account depolarizing errors in transmission and local two-qubit gate errors in each station during the Bell-state purification. A similar channel was analyzed in [14], where the authors found a biased error model arising from Bell state purification procedure. Second, we provide a detailed analysis of the non-local CNOT gate procedure, used to create entanglement between memory qubits in two stations. Considering the use of entanglement resources, local operations, measurement, and classical feedback, an error model of the non-local CNOT gate is provided. The resulting quantum channel has biased and correlated terms, with biased errors

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dominating. Here, we further show that non-symmetric CSS codes can be used to correct the correlated channel, and we propose a tailored code design to address the dominant biased errors.

In the 2G QR error model, different stations may suffer from different biased errors, which means that using different codes at the stations may lead to better performance. To comply with the non-local CNOT gate procedure, we need to keep the CNOT gate transversal for the pair of codes. Motivated by this, we provide sufficient and necessary conditions for CNOT gate transversality between different CSS codes. While similar cross-code operations have been explored in the context of lattice surgery and with the assistance of Pauli measurements [15]–[18], these methods provide alternative approach for implementing logical operators across different QECCs. However, the additional resource overhead introduced by such approaches makes them less compatible with near-term 2G quantum repeater systems. In this work, rather than addressing universal fault-tolerant cross-code operations, we focus specifically on using transversal physical gates to implement logical operations within 2G QR architectures. Considering the fact that the transversal CZ gate with local operations leads to a non-local CNOT gate, we also provide sufficient conditions for achieving a transversal non-local CZ gate. We then design a family of mirrored structure codes for correcting different biased errors between neighboring stations, and prove that these codes preserve the transversality of the logical CZ gate. It is known that a symmetric CSS code has Hadamard gate transversality. We verified this condition using our framework and show that the mapping matrix should fulfill specific conditions, which naturally arise from the commutation relation of the logical operations. We also provide an example of having both CZ and Hadamard gate transversalities using different codes at different stations, and compare it with the mirrored structure code design. We also show that the mapping of logical to physical qubits in the pair of codes will affect transversality, and identify the conditions on the mapping matrices that allow transversality. We provide examples of pairs of CSS codes fulfilling either CNOT or CZ transversality. In addition, we discuss the possibility of universal computation via distinct QECCs. We conclude that, with this approach and using the Clifford + $\mathbf{T}$ gate, it is not possible to achieve transversality and universality simultaneously.

Finally, we consider the rate of end-to-end entanglement generation in a multihop 2G QR quantum communication scenario. In a situation with limited quantum memory, the time cost of generating purified Bell pairs, the distance-dependent depolarizing errors happening on the fiber, and the errors happening in the non-local CNOT gate leads to an intricate connection between the number of hops, the overall communication distance, the gate error rate and the QECC used. We formulate an optimization problem for the length of a hop, given constraints on end-to-end fidelity and end-to-end entanglement generation rate.

With numerical simulation, we compare codes with CNOT and CZ transversality, show that in a range of parameters, CZ-transversal codes achieve better fidelity. Through numerical evaluation, we show that given constraints on the end-to-end entanglement generation rate and fidelity, the number of hops can be considerably reduced if mirrored structure codes are used instead of a pair of symmetric CSS codes. In particular for low gate error rate, we can reduce the number of hops to $1/3$ of the symmetric case.

This paper is organized as follows: Preliminaries about QECCs, 2G QRs and transversality can be found in Section II. Section III describes the detailed error model for 2G QRs. Based on this, two QECCs design are proposed in this section. In order to keep QRs protocol fault-tolerant, conditions on CSS codes which are CNOT or CZ transversal are discussed in Section IV. The effect of logical qubit mapping and universality is also discussed in this section. In Section V, we analyze the rate of entanglement generation for 2G QRs and set the optimization conditions for minimizing the number of repeater stations. Numerical simulation results are shown in Section VI, while Section VII concludes the paper.

II. Preliminaries

In this section, we provide the definition of the CSS codes and discuss the basic concepts of 2G quantum repeaters.

A. CSS Codes

Calderbank-Shor-Steane (CSS) quantum error correction codes are a special case of stabilizer codes. They can be constructed based on two classical linear codes. Considering two linear codes $\mathcal{C}_1(n,k_1,d_1)$, $\mathcal{C}_2(n,k_2,d_2)$, such that $\mathcal{C}_2^{\perp}\subset\mathcal{C}_1$, a quantum $[[n,k,d]]$ CSS code $\mathcal{Q}_{12}\triangleq \text{CSS}(\mathcal{C}_1,\mathcal{C}_2)$, is defined as a linear subspace of dimension 2^k in $\mathbb{C}^{2^n}$ with the orthonormal basis [11]:

$$|\boldsymbol{\psi}\rangle_L=\frac{1}{\sqrt{|\mathcal{C}_2^{\perp}|}}\sum_{\mathbf{y}\in\mathcal{C}_2^{\perp}}|\mathbf{x}+\mathbf{y}\rangle, \tag{1}$$

where $\boldsymbol{\psi}\in\mathbb{F}_2^k$ and $\mathbf{x}\in\mathcal{C}_1$ a representative of the coset of $\mathcal{C}_2^{\perp}$ in the quotient group $\mathcal{C}_1/\mathcal{C}_2^{\perp}$ corresponding to $\boldsymbol{\psi}$. Note that $k=k_1+k_2-n$ and the code minimum distance is $d=\min\left(d_1,d_2^{\perp}\right)$. In (1), we assume that the map between $\boldsymbol{\psi}$ and cosets of the group $\mathcal{C}_1/\mathcal{C}_2^{\perp}$ is linear, i.e., there exist an $k\times n$ binary *logical qubit mapping matrix* $\mathbf{A}$ such that $\mathbf{x}=\boldsymbol{\psi}\mathbf{A}$ Note that $|\boldsymbol{\psi}\rangle_L\in\mathbb{C}^{2^n}$ is the quantum state of n physical qubits corresponding to k logical qubits in the state $|\boldsymbol{\psi}\rangle\in\mathbb{C}^{2^k}$.

Consider the binary vectors $\mathbf{v}=[v_1,...,v_n]$. The standard basis vector of $\mathbb{C}^N$ with $N=2^n$ can be defined as $|\mathbf{v}\rangle=|v_1\rangle\otimes...\otimes|v_n\rangle\in\mathbb{C}^N$ where $|0\rangle\triangleq[1,0]^T$ and $|1\rangle\triangleq[0,1]^T$ and $\otimes$ denotes the Kronecker (Tensor) product.

Pauli matrices for a single qubit system are defined as

$$\mathbf{X}\triangleq\begin{bmatrix}0&1\\1&0\end{bmatrix},\ \mathbf{Z}\triangleq\begin{bmatrix}1&0\\0&-1\end{bmatrix}\ \mathbf{Y}\triangleq\begin{bmatrix}0&-i\\i&0\end{bmatrix}, \tag{2}$$

where $i\triangleq\sqrt{-1}$. Pauli errors acting on n qubits have the form $\mathbf{E}(\mathbf{a},\mathbf{b})=i^{\mathbf{ab}^T}\mathbf{D}(\mathbf{a},\mathbf{b})$, where $\mathbf{D}(\mathbf{a},\mathbf{b})=\mathbf{X}^{a_1}\mathbf{Z}^{b_1}\otimes...\otimes\mathbf{X}^{a_n}\mathbf{Z}^{b_n}$ and $\mathbf{a}=[a_1,...,a_n]$, $\mathbf{b}=[b_1,...,b_n]$ are binary vectors. We denote by γ the homomorphism defined by $\gamma\left(i^{\kappa}\mathbf{D}\left(\mathbf{a},\mathbf{b}\right)\right)=\left[\mathbf{a}^T,\mathbf{b}^T\right]$, $\kappa\in\{0,1,2,3\}$.

A $\mathcal{Q}_{12}\triangleq$ CSS($\mathcal{C}_1,\mathcal{C}_2$) code is an $[[n,k]]$ stabilizer code [11] whose $n-k$ generators correspond to binary vectors $\mathbf{g}_i$, $i=1,...,n-k$ in the form $[\mathbf{a},\mathbf{0}]$ or $[\mathbf{0},\mathbf{b}]$, $\mathbf{a},\mathbf{b}\in\mathbb{F}_2^n$ and $\mathbf{0}\in\mathbb{F}_2^n$ is the all zero vector. Combining vectors $\mathbf{a}$ and $\mathbf{b}$, we obtain parity check matrices $\mathbf{H}\left(\mathcal{C}_2\right)$ and $\mathbf{H}\left(\mathcal{C}_1\right)$ respectively. Putting these matrices together we obtain the matrix $\mathbf{G}^{\mathcal{Q}}$, the matrix of stabilizer generators of CSS($\mathcal{C}_1,\mathcal{C}_2$):

$$\mathbf{G}^{\mathcal{Q}}=\left[\begin{array}{c:c}\mathbf{H}\left(\mathcal{C}_2\right) & \mathbf{0}\\ \hdashline \mathbf{0} & \mathbf{H}\left(\mathcal{C}_1\right)\end{array}\right] \tag{3}$$

The dimension of $\mathbf{G}^{\mathcal{Q}}$ is $(n-k)\times 2n$. Thus, with n qubits and $n-k$ generators, $\mathcal{Q}_{12}$ encodes k logical qubits. $\mathcal{C}_1$ and $\mathcal{C}_2$ are used to decode Pauli X and Z errors separately. The quantum code rate is defined as $R=k/n$. Increasing the code length n, while keeping R constant, we improve the code performance. However, the encoding and decoding complexity, and the number of operation errors, also grow with the code length n [19]. In a real physical system, we need to select a suitable code, based on the demand for logical fidelity and the accuracy of quantum operations.

Given $n-k$ stabilizer generators, we can define logical Pauli operators that are linearly independent and commute with all the generators. The Pauli operators can be represented as binary vectors. The $2n$ binary space can be decomposed into $2n$ independent basis vectors. In addition to the $n-k$ linear independent vectors given by the stabilizer, there exists $n+k$ linearly independent vectors. The logical Paulis also need to commute with the $n-k$ generators. We can thus find $n+k-(n-k)=2k$ basis vectors which give rise to the group $\mathbf{U}_L$ of logical Pauli operators with cardinality $|\mathbf{U}_L|=4^k$. Thus, with k logical qubits, we have 4^k different logical Pauli operators.

B. Second generation Quantum Repeaters

Three generations of QRs exist based on different protocols, with 2G QRs likely to be realizable in the near future [4]. In contrast to the first generation, 2G QRs use QECC to suppress the errors, which leads to a more efficient way to mitigate transmission errors in quantum communication. Third generation repeaters provide a higher communication rate than 2G ones. However, they demand the transmission of many physical qubits to reduce the errors [12]. Second generation repeaters, with the process of Bell state purification, have a lower rate but they do not require large-scale QECCs. Thus, for near-future quantum registers with some hundreds of qubits, 2G QRs may be preferable.

In brief, the 2G QR protocol begins by establishing and purifying raw photonic Bell states between neighboring stations. These purified states are used to generate logical entanglements between qubits stored in the quantum memories at each station, enabling the correction of errors arising from local operations and memory decoherence. Finally, entanglement swapping is performed to establish end-to-end logical entanglements. More details are provided as follows:

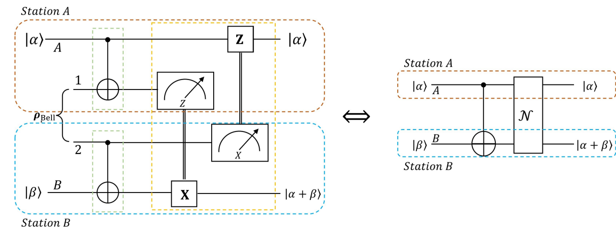

Fig. 1. A non-local CNOT gate between stations A and B is applied using a Bell pair with two local CNOT gates. State $|\alpha\rangle$ and the first qubit of the Bell pair located in station A, while $|\beta\rangle$ and the second qubit of the Bell pair located in station B.

1) The logical states $q_A=|+...+\rangle_L$, where $|+\rangle=\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$, and $q_B=|0...0\rangle_L$ with k qubits are encoded into n physical qubits p_A and p_B in neighboring stations A and B.
2) Generate Bell-pair qubits $b_1^1b_2^1$ and $b_1^2b_2^2$... are in state $\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)$ in station A and transmit $b_2^t, t=1,2,...$ to station B through optic fiber.
3) Some of the transmitted qubits are lost on the channel. Heralded entanglement generation is used against erasure errors.
4) Among the qubits that station B has received successfully, Bell state purification is applied as follows [20]. CNOT gates are executed at A and B between qubits of two Bell states $b_1^tb_1^{t'}$ and $b_2^tb_2^{t'}$ with b_1^t and b_2^t as the control qubits, respectively. The target qubits are measured, the outcome indicating whether purification succeeded. In this step, most Bell pairs are measured out and only a fraction of them will remain, with high fidelity.
5) A non-local CNOT gate is realized between p_A and p_B. CNOT gates are conducted at the stations from the physical qubits in memory to the Bell-pair qubit; from p_A to b_1^t and b_2^t to p_B. The Bell-pair qubits $b_1^t\,b_2^t$ are measured. According to the measurement feedback, Pauli operators $\mathbf{Z}$ and $\mathbf{X}$ are applied on p_A and p_B separately. The non-local CNOT gate circuit is shown in Fig. 1.
6) Transversal action of n non-local CNOT gates on n paris of physical qubits p_A and p_B of step 5 leads to entanglement between logical qubits q_A and q_B. The resulting logical state between station A and B is $|00\rangle_L+|11\rangle_L$.
7) QECC is used in both stations to correct errors.
8) Entanglement swapping is performed at intermediate stations to achieve end-to-end entanglement [21]. Local CNOT gates are applied between code blocks related to two different hops. The measurements are performed simultaneously at all intermediate stations. The outcomes are sent to the outermost stations over the classical communication channel. These execute logical Pauli operators corresponding to the aggregate measurements to restore an end-to-end logical Bell state.

In 2G QRs, *QECCs with CNOT transversality* are used.

This guarantees entanglement generation between q_A and q_B in step 6. Note that if a QECC with $k > 1$ is used, the transversal non-local CNOT gate would create entanglement between k pairs of logical qubits at station A and B.

The created end-to-end entanglement can be used for key generation, or for quantum communication using quantum teleportation. Irrespective of the intended use, the essential parameters governing system performance are the end-to-end Entanglement Generation Rate (EGR), and the fidelity of the created logical Bell state.

C. Transversality

QECCs encode many physical qubits into logical qubits and protect logical information from transmission and system operation noise by using redundancy. In 2G QRs systems, QECCs are used in each station to correct system errors. Logical entanglement is established between different code blocks, thus it is important to keep the process fault-tolerant to prevent error propagation between different qubits. Transversal operators provide the easiest way to achieve fault-tolerant quantum circuits, i.e., preventing error propagation between different code blocks [22]. Transversality means that applying operations to the physical qubits in parallel leads to the same operations in the logical space. For an n-qubit error correction code we can thus write the logical operator as

$$\mathbf{U}_L = \bigotimes_{i=1}^{n} \mathbf{U}_i \,, \tag{4}$$

where $\mathbf{U}_i$ indicates the quantum operator $\mathbf{U}$ acting on qubit i.

Stabilizer codes are a family of widely investigated QECCs. In stabilizer codes, the information is stabilized by code generators, which are Pauli operators, into the stabilizer space [23]. CSS codes are a special case of stabilizer codes wherein the generators are pure $\mathbf{X}$ and $\mathbf{Z}$ Pauli operators and the code is constructed by two classical linear codes [24]. The transversality of the CSS codes has been investigated in [25]. It was shown that all CSS codes have Pauli and CNOT gate transversality and that self-orthogonal CSS codes are also Hadamard and CZ transversal.

III. Errors and Error Correction for 2G QRs

In this section, we analyze the error channel originating from the 2G QR procedure, and discuss two QECC designs that outperform conventional designs in this situation.

First, we consider two sources for errors: transmission errors and operation errors. Assuming the 2G QR protocol completes within the coherence time of quantum memory, we neglect quantum memory errors. Then we model how these basic errors propagate through key procedures: Bell state purification and non-local CNOT gate implementation. The fidelity relation between purified Bell state and initial input states through iteration of purification procedure has been well studied [26], [27]. As for non-local CNOT gate, we show this procedure will lead to a special error model which includes both biased and correlated errors. On the basis of the resulting error model, we show how to use different QECCs to tailor these different error models.

A. Transmission error

The transmission of Bell pairs between neighboring stations through the optical fiber is susceptible to erasure and depolarizing errors that impact the photonic qubits. The crucial aspect of transmission errors is that the error probability depends on the distance between the stations. We model erasure error probability by [28]

$$e_0 = 1 - 10^{-\alpha L_0/10} \tag{5}$$

where L_0 is the distance between neighboring stations, and α is the attenuation coefficient which depends on the medium that the qubits have been transmitted through. It is important to note that attenuation significantly influences the EGR. Given the erasure probability e_0, the transmitter needs on average $\frac{1}{1-e_0}$ attempts for a successful transmission of one photonic qubit.

In addition to erasures, we consider depolarizing errors which model the errors resulting from the interaction of qubits with the environment. The depolarizing error rate also increases exponentially with the transmission distance. It can be modeled as [29]

$$\varepsilon_{de} = 1 - e^{-\beta L_0} \tag{6}$$

where β is the noise coefficient, which is a constant parameter depending on the quality of the medium.

The quantum transmission channel can thus be written as:

$$\mathcal{N}_{tr}(\boldsymbol{\rho}) = (1-\varepsilon_{de}-e_0)[\mathbf{I}](\boldsymbol{\rho}) + \frac{\varepsilon_{de}}{3}\left([\mathbf{X}] + [\mathbf{Y}] + [\mathbf{Z}]\right)(\boldsymbol{\rho}) \tag{7}$$

Here $[\mathbf{U}](\boldsymbol{\rho}) = \mathbf{U}\,(\boldsymbol{\rho})\mathbf{U}^{\dagger}$ represents the conjugate action of $\mathbf{U}$ on the density matrix $\boldsymbol{\rho}$. Note that with probability e_0 an erasure error happens. The qubit then vanishes on the channel and its density matrix at channel output is $\boldsymbol{\rho} = \mathbf{0}$.

B. Operation error

We assume that all physical qubits and quantum operators are prone to depolarizing errors. Generally speaking, single-qubit gate errors and quantum memory errors are significantly less prevalent than two-qubit gate errors. For performance evaluation, we assume that these two types of errors are two orders of magnitude smaller than two-qubit gate errors [30]. Since the main two-qubit gate used in 2G QR is the CNOT gate, we focus on the CNOT gate error model. We use the depolarizing model for CNOT gate errors [27]:

$$\mathcal{N}_{\text{CNOT}}(\boldsymbol{\rho}) = (1 - f_{\text{gate}})\,[\mathbf{II}](\boldsymbol{\rho}) + \frac{f_{\text{gate}}}{16}\sum_{i,j=1}^{4}[\mathbf{P}_c^i\mathbf{P}_t^j](\boldsymbol{\rho})\,, \tag{8}$$

where $\mathbf{P}_c^i$, $\mathbf{P}_t^j$ denote control and target qubit Pauli operators in the set $\{\mathbf{I}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$, respectively, and f_{gate} is the CNOT gate error rate.

C. Error model for Bell state purification

Upon reception of the noisy Bell state, we employ a Bell state purification procedure at both stations [26]. For K layers of purification, at least 2^K pairs of Bell states and 2^K-1 CNOT gates are needed. When more purification layers are used, the final Bell state achieves higher fidelity, as long as the initial fidelity is higher than 50% [28]. The Bell states go through a Pauli error channel, which leads to a Pauli error model for the Bell state:

$$\boldsymbol{\rho}_{\text{Bell}} = f_0\boldsymbol{\rho}_{00} + f_1\boldsymbol{\rho}_{01} + f_2\boldsymbol{\rho}_{10} + f_3\boldsymbol{\rho}_{11}, \tag{9}$$

where $f_0 = 1 - f_1 - f_2 - f_3$ is the fidelity, and

$$\begin{aligned}
\boldsymbol{\rho}_{00} &= \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)\\
\boldsymbol{\rho}_{01} &= \frac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|)\\
\boldsymbol{\rho}_{10} &= \frac{1}{2}(|00\rangle - |11\rangle)(\langle 00| - \langle 11|)\\
\boldsymbol{\rho}_{11} &= \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)\,.
\end{aligned}$$

The ideal Bell state is $\boldsymbol{\rho}_{00}$, and f_1, f_2 and f_3 are the probability of $\boldsymbol{\rho}_{01}$, $\boldsymbol{\rho}_{10}$, and $\boldsymbol{\rho}_{11}$, respectively. If we use $\boldsymbol{\rho}_{00}$ as ideal state, other states can be written as a Pauli error acting on the ideal state:

$$\boldsymbol{\rho}_{01} = \mathbf{X}\,\boldsymbol{\rho}_{00}\,\mathbf{X},\;\; \boldsymbol{\rho}_{10} = \mathbf{Z}\,\boldsymbol{\rho}_{00}\,\mathbf{Z},\;\; \boldsymbol{\rho}_{11} = \mathbf{ZX}\,\boldsymbol{\rho}_{00}\,\mathbf{XZ}\,.$$

Note that the two Pauli operators in $\mathbf{XZ}$ term can act on any of Bell pair qubits, since Bell state is the eigen state of $\mathbf{X_1X_2}$ and $\mathbf{Z_1Z_2}$ operations.

Since the transmitted photonic Bell pairs are corrupted by the distance-related depolarizing channel (7), the input states of first-level purification have error probabilities

$$f_1 = f_2 = f_3 = \varepsilon_{de}/3 \tag{10}$$

which are distance-related. After each level purification, new error probabilities can be calculated [26]:

$$\begin{cases} f_0 = f_0'^2 + f_2'^2\\ f_1 = 2f_0'f_1'\\ f_2 = 2f_2'f_3'\\ f_3 = f_1'^2 + f_3'^2 \end{cases} \tag{11}$$

Here f_i and f_i' are probabilities of the purified and input states. Since the error probabilities should be small, such that $f_0 >> f_1 + f_2 + f_3$, for the purified state we have $f_1 > f_2 + f_3$.

The purification success probability p_s is [26]

$$p_s = (f_0' + f_2')^2 + (f_1' + f_3')^2 = (1 - f_1' - f_3')^2 + (f_1' + f_3')^2 \tag{12}$$

where we used $\sum_i f_i' = 1$ for the second form. Only two types of Pauli errors affect the measurement outcome and the success probability. The biased error probabilities of purified Bell states were studied in [14], where a different Bell-state purification procedure is used, this leads to a different, but also biased error model for the output states.

D. Error model for non-local CNOT gate

In 2G QRs, logical entanglement between neighboring stations is established by transversal non-local CNOT gates. With the help of a shared purified Bell state, the non-local CNOT gate can be realized with high-fidelity from a physical qubit in station A to the corresponding physical qubit in station B. The resulting quantum channel arising from these operations was reported in [31]. Here we provide the underlying analysis.

The purified Bell state as in (9) is used as input state to realize a non-local CNOT gate. The circuit of the non-local CNOT gate is shown in Fig. 1. This circuit is equivalent to a perfect CNOT gate with a noisy channel $\mathcal{N}$. If we neglect the local gate error, the parameters of $\mathcal{N}$ come from the purified state $\boldsymbol{\rho}_{\text{pur}}$.

First, we go through the situation with ideal input $\boldsymbol{\rho}_{00}$, to understand the procedure. The non-local state after the two CNOT gates before measurement can be described as

$$\begin{aligned}
|\boldsymbol{\Psi}_{\text{ideal}}\rangle &= \boldsymbol{\Lambda}_{A,1}\boldsymbol{\Lambda}_{2,B}|\alpha\rangle_A\,\frac{1}{\sqrt{2}}(|00\rangle_{12} + |11\rangle_{12})|\beta\rangle_B \qquad (13)\\
&= \alpha_0|000\beta\rangle + \alpha_0|011\beta'\rangle + \alpha_1|110\beta\rangle + \alpha_1|101\beta'\rangle\,,
\end{aligned}$$

where the subscript i indicates the ith qubit, $\boldsymbol{\Lambda}_{i,j}$ means CNOT gate from qubit i to j, with matrix

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix},$$

the first input state is $|\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, and $|\beta'\rangle = \mathbf{X}|\beta\rangle$. Next, we conduct Z- and X-basis measurements on the 2nd and 3rd qubits. Each measurement has two outcomes with the same probability, which means that the final outcome is uniformly random. Using $\mathbf{M}_{Z,X}$ to represent the two measurement results, the state after the measurements are:

$$\begin{cases} \mathbf{M}_{Z,X} = (+1,+1): & \alpha_0|00+\beta\rangle + \alpha_1|10+\beta'\rangle\\ \mathbf{M}_{Z,X} = (+1,-1): & \alpha_0|00-\beta\rangle - \alpha_1|10-\beta'\rangle\\ \mathbf{M}_{Z,X} = (-1,+1): & \alpha_0|01+\beta'\rangle + \alpha_1|11+\beta\rangle\\ \mathbf{M}_{Z,X} = (-1,-1): & \alpha_0|01-\beta'\rangle - \alpha_1|11-\beta'\rangle \end{cases} \tag{14}$$

For $\mathbf{M}_{Z,X} = (+1,+1)$, the final state is the same as if a CNOT gate were executed; $\boldsymbol{\Lambda}_{\alpha\beta}|\alpha\rangle\otimes|\beta\rangle = \alpha_0|0\beta\rangle + \alpha_1|1\beta'\rangle$. For the other measurement outcomes, classical communication of the measurement results followed by single qubit Pauli operators at different stations can recover the ideal state. For example, if we have $\mathbf{M}_{Z,X} = (+1,-1)$, then a Pauli $\mathbf{Z}$ operator at Station A will recover the state to ideal state, $\mathbf{Z}(\alpha_0|0\beta\rangle - \alpha_1|1\beta'\rangle) = \alpha_0|0\beta\rangle + \alpha_1|1\beta'\rangle$.

The action of the non-local CNOT gate on erroneous purified states in (9) can be analyzed in a similar manner. Due to the specific form of the noisy Bell state, only three types of Pauli errors have to be considered. E.g., instead of an ideal Bell state, assume that $\boldsymbol{\rho}_{01}$ is the input of this circuit. Since the ideal Bell state $\boldsymbol{\rho}_{00}$ is an eigenstate of $\mathbf{XX}$ operator, this can arise in two ways—-a Pauli $\mathbf{X}$ error happens either on the first or on the second qubit. First assume that it happened on the first Bell-state qubit, in station A. This error goes through the upper CNOT gate and affects the Z-basis measurement outcome. Because a recovery Pauli $\mathbf{X}$ operation is acting at Station B according to the measurement, the resulting outcome will have a Pauli $\mathbf{X}$ error on the physical qubit in station B. If an $\mathbf{X}$ error occur on the second Bell state qubit, after the lower CNOT gate, an $\mathbf{X}$ error propagates to the physical qubit in station B. The X-basis measurement on the second Bell pair qubit at Station B is robust to $\mathbf{X}$ errors, and will not be affected. This procedure thus also results in an $\mathbf{X}_B$ error.

We can similarly verify that a $\mathbf{Z}$ error, corresponding to $\boldsymbol{\rho}_{10}$ as the input state, will lead to a $\mathbf{Z}$ error in station A. When the first Bell state qubit has a $\mathbf{Z}$ error, it will propagate through the upper CNOT gate to the physical qubit in station A. A Z-basis measurement commutes with a $\mathbf{Z}$ error, thus no error is propagated to station B, and the final state has a $\mathbf{Z}_A$ error. In the same way we can see a $\mathbf{Z}$ error on the second Bell state qubit will affect the X-basis measurement and also leads to a $\mathbf{Z}_A$ error.

Finally, the input state $\boldsymbol{\rho}_{11}$ arises when we have an $\mathbf{X}$ and a $\mathbf{Z}$ error on the qubits in the Bell-pair. There are four equivalent alternatives for this. These result in the product error $\mathbf{Z}_A\mathbf{X}_B$ for the non-local CNOT gate. We can verify this by writing out the whole procedure. For the case when the two errors happen on the second qubit of the Bell pair, we have

$$\begin{aligned}
&\boldsymbol{\Lambda}_{A,1}\boldsymbol{\Lambda}_{2,B}|\alpha\rangle_A\,\mathbf{Z}_1\mathbf{X}_1\,\frac{1}{\sqrt{2}}(|00\rangle_{12}+|11\rangle_{12})|\beta\rangle_B\\
&\quad=\mathbf{Z}_A\mathbf{Z}_1\mathbf{X}_1\boldsymbol{\Lambda}_{A,1}\boldsymbol{\Lambda}_{2,B}|\alpha\rangle_A\,\frac{1}{\sqrt{2}}\left(|00\rangle_{12}+|11\rangle_{12}\right)|\beta\rangle_B\\
&\quad=\mathbf{Z}_A\mathbf{Z}_1\mathbf{X}_2|\boldsymbol{\Psi}_{\text{ideal}}\rangle
\end{aligned}\tag{15}$$

in terms of the ideal state in (13). Feedback applied according to measurement outcomes, as in (14). When conducting measurements on the noisy state in (15), ancillary state qubits 1 and 2 are measured out. The Pauli $\mathbf{X}$ error on the first Bell pair qubit anti-commutes with Z-basis measurement and flips the outcome. This leads to an additional $\mathbf{X}$ operator acting on $|\beta\rangle$. In the end, there is $\mathbf{Z}_A\mathbf{X}_B$ error on the final state.

As a consequence of the analysis above, the error channel $\mathcal{N}$ for a non-local CNOT gates becomes:

$$\begin{aligned}\mathcal{N}(\boldsymbol{\rho}) \;=\;& f_0[\mathbf{I}_A\mathbf{I}_B](\boldsymbol{\rho})\;+\;f_1[\mathbf{Z}_A](\boldsymbol{\rho})\\ &+\;f_2[\mathbf{X}_B](\boldsymbol{\rho})\;+\;f_3[\mathbf{Z}_A\mathbf{X}_B](\boldsymbol{\rho})\end{aligned}\tag{16}$$

Here, hardware qubits in stations A and B are indicated by corresponding subscripts.

In this quantum channel, there are two characteristic parts. There are biased errors $[\mathbf{Z}_A]$ and $[\mathbf{X}_B]$—different types of Pauli errors happen in each station. In addition, there are correlated errors $[\mathbf{Z}_A\mathbf{X}_B]$ which occur in both stations simultaneously. For instance, the probability that the ith qubit in station A incurs a Pauli $\mathbf{Z}$ error is f_1+f_3. When this happens, the conditional probability that the ith qubit in station B will experience a Pauli $\mathbf{X}$ error simultaneously is $f_3/(f_1+f_3)$.

Thus, in 2G QRs, in addition to gate errors, there exist biased and correlated errors introduced by Bell state purification that have to be taken into account. The biased and correlated errors grow as the distance between the neighboring stations increases, and are significant when the gate error probability is sufficiently low.

From the recursion (11) of the purification errors, it follows that $f_1 > f_2, f_3$. Thus the dominating term in the system is the biased error, specifically the biased $[\mathbf{Z}_A]$ happening in Station A. We can also apply single-qubit gates to the purified Bell pair, which can be described as (9), to exchange the values of f_1, f_2 and f_3, thereby adjusting the dominant error type. For example, applying the $\mathbf{S}\otimes\mathbf{S}^\dagger$ gate allows us to swap the values of f_1 and f_3:

$$\mathbf{S}\otimes\mathbf{S}^\dagger\boldsymbol{\rho}_{\text{Bell}}\mathbf{S}^\dagger\otimes\mathbf{S}=f_0\boldsymbol{\rho}_{00}+f_1\boldsymbol{\rho}_{11}+f_2\boldsymbol{\rho}_{10}+f_3\boldsymbol{\rho}_{01},\tag{17}$$

where $\mathbf{S}=\begin{bmatrix}1&0\\0&i\end{bmatrix}$, and $\mathbf{S}\mathbf{X}\mathbf{S}^\dagger=\mathbf{Y}$. Using this modified Bell pair as input Bell pair of non-local CNOT gate procedure, we can change error channel (16) into $\mathcal{N}'$:

$$\begin{aligned}\mathcal{N}'(\boldsymbol{\rho})=\;& f_0[\mathbf{I}_A\mathbf{I}_B](\boldsymbol{\rho})\;+\;f_3[\mathbf{Z}_A](\boldsymbol{\rho})\\ &+\;f_2[\mathbf{X}_B](\boldsymbol{\rho})\;+\;f_1[\mathbf{Z}_A\mathbf{X}_B](\boldsymbol{\rho})\end{aligned}\tag{18}$$

with $f_1 > f_2, f_3$. In this case, the dominating error is correlated error $[\mathbf{Z}_A\mathbf{X}_B]$. Based on these 2G QR error models, optimized QECC protocols can be designed.

We neglected the local gate error when deriving the non-local CNOT gate error model (16), which is independent of physical gate error rate f_{gate}. To complement the error model, we consider depolarizing errors for two-qubit gates as (8), resulting in:

$$\begin{aligned}\mathcal{N}_O(\boldsymbol{\rho})&=\mathcal{N}^2_{\text{CNOT}}(\mathcal{N}(\boldsymbol{\rho}))\\
&\approx(1-2f_{\text{gate}})\mathcal{N}(\boldsymbol{\rho})+\frac{2f_{\text{gate}}}{16}\sum_{i,j=1}^{4}[\mathbf{P}_c^i\mathbf{P}_t^j]\mathcal{N}(\boldsymbol{\rho}),\end{aligned}\tag{19}$$

where $\mathcal{N}^2_{\text{CNOT}}$ stands for the error channel of two local CNOT gates used in non-local CNOT gate circuit.

E. Non-symmetric CSS Codes in Correlated Error Channels

It is important to note that in a 2G QR protocol, two blocks of logical qubits are mapped to physical qubits, one at station A, one at B. The objective is to create entanglement between stations A and B. Thus, in the encoding phase, there is no entanglement and the encoding has to be performed separately at stations A and B. However, decoding may be performed jointly at the stations, subject to an exchange of classical syndrome bits.

To achieve better performance, we need to design a proper QECC according to the dominant error in the system. As we discussed in the last section, the purified Bell state model (16) can be modified into correlated-dominated error model (18), where $[\mathbf{Z}_A\mathbf{X}_B]$ is dominated term. With such a channel, we can first assume $f_1 \gg f_2+f_3$, concentrate an error model with only correlated errors and neglect other errors:

$$\mathcal{N}_{co}=(1-f)[\mathbf{II}]+f[\mathbf{Z}^A\mathbf{X}^B],\tag{20}$$

where $\mathbf{Z}^A$ and $\mathbf{X}^B$ are Pauli operators acting on the qubit in stations A and B, respectively. This means that only $\mathbf{Z}$ errors happen in station A, and only $\mathbf{X}$ errors in station B, and these happen on the same physical qubits.

Now we consider the situation where an $[[n,k,d]]$ stabilizer code $\mathcal{Q}$ with minimum distance $d=2t+1$ is used in stations A and B to encode the logical $|+\rangle$ and $|0\rangle$ states, respectively. We consider joint decoding at the two stations. That is, decoding is performed for the joint code $\mathcal{Q}\otimes\mathcal{Q}$.

For an error event e with weigh $k>t$, there exists an error with weight $l\leq t$ which has the same syndrome as e. If we have a CSS code, a Pauli X-error event $e_{X1}=\prod_{i\in\mathcal{S}_1}\mathbf{X}_i$ on qubits in set $\mathcal{S}_1$ with $|\mathcal{S}_1|>t$ has the same syndrome as an error event $e_{X2}=\prod_{i\in\mathcal{S}_2}\mathbf{X}_i$ at positions $\mathcal{S}_2$ with $|\mathcal{S}_2|\leq t$.

The CSS code $\mathcal{Q}_{12}$ may be symmetric, with $\mathcal{C}_1=\mathcal{C}_2=\mathcal{C}$, or non-symmetric. For a symmetric code, we have the same situation for Pauli $\mathbf{Z}$ errors as for $\mathbf{X}$ errors. Thus $\mathbf{Z}$ error $e_{Z1}=\prod_{i\in\mathcal{S}_1}\mathbf{Z}_i$, has the same syndrome as $\mathbf{Z}$ error $e_{Z2}=\prod_{i\in\mathcal{S}_2}\mathbf{Z}_i$, and it cannot be corrected. As a consequence of this we have

Proposition 1. *Consider a non-symmetric and a symmetric CSS code with the same minimum distance. They are used at stations A and B in quantum channel* (20) *with correlated* $\mathbf{ZX}$ *errors. With joint decoding at the stations, the non-symmetric CSS code has smaller error probability than the symmetric one.*

Proof. Consider Pauli error $e_{AB}=e_{X1}^A\otimes e_{Z1}^B=\prod_{i\in\mathcal{S}_1}\mathbf{X}_i^A\mathbf{Z}_i^B$; with $|\mathcal{S}_1|>t$ in neighboring stations A and B. With the error model (20), the probability of such error is $P(e_{AB})=f^{w(e_{X1})}$, where $w(e_{X1}^A)$ means the weight of Pauli error e_{X1}^A. According to the definition of a symmetric CSS code, there exists an error event $e_3=e_{X2}^A\otimes e_{Z2}^B=\prod_{i\in\mathcal{S}_2}\mathbf{X}_i^A\mathbf{Z}_i^B$, with $|\mathcal{S}_2|\leq t$, which has the same syndrome as e_{AB}. The probability of this event is $P(e_3)=f^{w(e_{X2}^A)}$. Since e_{X2}^A is a low weight error, $P(e_3)>P(e_{AB})$. In the error correction procedure, we select the error with the highest probability among the errors with the same syndrome. Accordingly, using symmetric CSS codes, we cannot recover e_{AB} errors. Thus the decoding error probability will be $O\left(P(e_{AB})\right)$.

However, for non-symmetric CSS code, the error event with the same syndrome would be $e_3'=e_{X2}^A\otimes e_{Z3}^B=\prod_{i\in\mathcal{S}_2}\prod_{j\in\mathcal{S}_3}\mathbf{X}_i^A\mathbf{Z}_j^B$ with $\mathcal{S}_2\neq\mathcal{S}_3$. That is, $\mathbf{Z}$ and $\mathbf{X}$ errors would happen in different positions in stations A and B. In error model (20), the probability of this type of error is $P(e_3')=0$. Thus with a non-symmetric CSS code, we can correct the e_{AB} error, and the decoding error rate would be lower than $O(P(e_{AB}))$, which is better than symmetric CSS code. □

F. Different CSS Codes at Stations A and B for Biased and Correlated Channels

Now we consider the more general error model (16), which contains both biased and correlated errors between two stations. According to this model, in station A, physical qubits are more prone to Pauli $\mathbf{Z}$ errors, while in station B, Pauli $\mathbf{X}$ errors are more prevalent. It is possible to use different codes at stations A and B, i.e., instead of the overall code $\mathcal{Q}\otimes\mathcal{Q}$, we may use $\mathcal{Q}_A\otimes\mathcal{Q}_B$. From the perspective of creating non-local entanglement, the overall QECC of interest is $\mathcal{Q}_A\otimes\mathcal{Q}_B$, there is no reason to limit oneself to $\mathcal{Q}\otimes\mathcal{Q}$.

For biased errors of type (16), it is possible to use a non-symmetric code in station A that can correct more $\mathbf{Z}$ errors, and a different non-symmetric code in station B that can correct more $\mathbf{X}$ errors. This approach may offer improved performance compared to using identical codes at both stations. For correcting biased errors, joint decoding is not strictly necessary because the biased errors are local errors. Although joint decoding can provide some additional benefit, the associated overhead in classical communication may negatively impact the entanglement generation rate.

However, with this approach, the transversality of the logical CNOT gates needed to create logical entanglement between stations A and B cannot be guaranteed in general. Using transversal logical gates is the most common way to realize fault-tolerant quantum circuits. In the next section, we shall address the problem of designing different codes $\mathcal{Q}_A$ and $\mathcal{Q}_B$ for the two stations, such that transversality of two-qubit gates is fulfilled.

IV. Transversality with Different Codes at Stations A and B

In this section, we study the restrictions for achieving transversality across two different codes with the same k and n. We assume that the QECCs used in stations A and B are CSS codes and investigate the transversality of logical CNOT and CZ gates between neighboring stations. We find a constructive method of creating a family of CZ transversal codes.

Note that here we only consider *parallel* logical CNOT/CZ. Each logical qubit of $\mathcal{Q}_A$ is paired with a logical qubit of $\mathcal{Q}_B$, and a transversal action of CNOT/CZ on the physical qubits of $\mathcal{Q}_A$ and $\mathcal{Q}_B$ realize a CNOT/CZ acting on these pairs of logical qubits in parallel.

A. CNOT Transversality

Assume that we have two codes $\text{CSS}(\mathcal{C}_1,\mathcal{C}_2)$ and $\text{CSS}(\mathcal{C}_3,\mathcal{C}_4)$ in stations A and B, respectively. Our objective is to find sufficient conditions for having transversal CNOT. Let $\mathbf{G}_2^\perp$ and $\mathbf{G}_4^\perp$ be the generator matrices of $\mathcal{C}_2^\perp$ and $\mathcal{C}_4^\perp$, respectively. The generator matrices of $\mathcal{C}_1$ and $\mathcal{C}_3$ can then be written as

$$\mathbf{G}_1=\begin{bmatrix}\mathbf{G}_2^\perp\\ \mathbf{A}\end{bmatrix},\qquad \mathbf{G}_3=\begin{bmatrix}\mathbf{G}_4^\perp\\ \mathbf{B}\end{bmatrix}, \tag{21}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $k\times n$ binary matrices of rank k. These matrices map n dimensional physical qubit space to k dimensional logical space. For row space of mapping matrix $\mathbf{A}$, you can add any row belongs to $\mathbf{G}_2^\perp$ without changing the encoding states. But adding rows in $\mathbf{A}$ itself will change the mapping matrix and leads to the different encoding.

Let $\boldsymbol{\psi}^A, \boldsymbol{\psi}^B \in \mathbb{F}_2^k$. The code vectors of CSS$(\mathcal{C}_1, \mathcal{C}_2)$ and CSS$(\mathcal{C}_3, \mathcal{C}_4)$ corresponding to logical qubits in the states $|\boldsymbol{\psi}^A\rangle$ and $|\boldsymbol{\psi}^B\rangle$ are

$$|\boldsymbol{\psi}^A\rangle_L = \frac{1}{\sqrt{|\mathcal{C}_2^\perp|}} \sum_{\mathbf{y}\in C_2^\perp} |\mathbf{x}^A + \mathbf{y}\rangle \tag{22}$$

$$|\boldsymbol{\psi}^B\rangle_L = \frac{1}{\sqrt{|\mathcal{C}_4^\perp|}} \sum_{\mathbf{z}\in C_4^\perp} |\mathbf{x}^B + \mathbf{z}\rangle, \tag{23}$$

where $\mathbf{x}^A = \boldsymbol{\psi}^A \mathbf{A}$ and $\mathbf{x}^B = \boldsymbol{\psi}^B \mathbf{B}$. We have [32]

Theorem 1. *Consider CSS$(\mathcal{C}_1, \mathcal{C}_2)$ and CSS$(\mathcal{C}_3, \mathcal{C}_4)$ codes in stations A and B, respectively, with generators given by* (21)*. The logical CNOT gate between stations A and B is transversal iff*

$$\mathcal{C}_2^\perp \subset \mathcal{C}_4^\perp \tag{24}$$

$$\mathcal{C}_1/\mathcal{C}_2^\perp \cong \mathcal{C}_3/\mathcal{C}_4^\perp \tag{25}$$

Proof. For showing transversality, we need to show that applying CNOT gates to k pairs of logical qubits in the states $|\boldsymbol{\psi}^A\rangle_L$ and $|\boldsymbol{\psi}^B\rangle_L$, and then encode the results into code vectors of CSS$(\mathcal{C}_1, \mathcal{C}_2)$ and CSS$(\mathcal{C}_3, \mathcal{C}_4)$ gives the same result as first encoding logical qubits and then applying CNOT gates to n pairs of the physical qubits. If we first apply CNOT gates to logical qubits and then encoding into code vectors of CSS$(\mathcal{C}_1, \mathcal{C}_2)$ and CSS$(\mathcal{C}_3, \mathcal{C}_4)$, we get

$$\boldsymbol{\Lambda}_{AB}^L \left(|\boldsymbol{\psi}^A\rangle_L \otimes |\boldsymbol{\psi}^B\rangle_L\right) \triangleq |\boldsymbol{\psi}^A\rangle_L \otimes |\boldsymbol{\psi}^A \oplus \boldsymbol{\psi}^B\rangle_L\,. \tag{26}$$

Using (21) and (22) leads to

$$|\boldsymbol{\psi}^A \oplus \boldsymbol{\psi}^B\rangle_L = \frac{1}{\sqrt{|\mathcal{C}_4^\perp|}} \sum_{\mathbf{y}\in C_4^\perp} |\left(\boldsymbol{\psi}^A + \boldsymbol{\psi}^B\right)\mathbf{B} + \mathbf{y}\rangle. \tag{27}$$

Next, by first encoding logical qubits in the states $|\boldsymbol{\psi}^A\rangle$ and $|\boldsymbol{\psi}^B\rangle$ and then applying CNOT gates to the n pairs of physical qubits, we get:

$$\begin{aligned}
&\boldsymbol{\Lambda}_{AB}^P \left(|\boldsymbol{\psi}^A\rangle_L \otimes |\boldsymbol{\psi}^B\rangle_L\right) \\
&= \frac{1}{\sqrt{|\mathcal{C}_2^\perp|\,|\mathcal{C}_4^\perp|}} \sum_{\mathbf{y}\in\mathcal{C}_2^\perp,\mathbf{z}\in\mathcal{C}_4^\perp} |\mathbf{x}^A + \mathbf{y}\rangle \otimes |\mathbf{x}^A + \mathbf{x}^B + \mathbf{y} + \mathbf{z}\rangle \\
&\overset{(a)}{=} \frac{1}{\sqrt{|\mathcal{C}_2^\perp|\,|\mathcal{C}_4^\perp|}} \sum_{\mathbf{y}\in\mathcal{C}_2^\perp,\mathbf{z}'\in\mathcal{C}_4^\perp} |\mathbf{x}^A + \mathbf{y}\rangle \otimes |\mathbf{x}^A + \mathbf{x}^B + \mathbf{z}'\rangle
\end{aligned} \tag{28}$$

where (a), $\mathbf{y}$ is absorbed into the sum over $\mathbf{z}$,is true iff $\mathcal{C}_2^\perp \subset \mathcal{C}_4^\perp$, or equivalently $\mathcal{C}_4 \subset \mathcal{C}_2$.

For achieving the transversality, (26) has to equal (28) for all $\boldsymbol{\psi}^A$ and $\boldsymbol{\psi}^B$. This is possible if and only if $\mathbf{A} = \mathbf{B}$. Note that $\mathbf{A} = \mathbf{B}$ also means that $\mathcal{C}_1/\mathcal{C}_2^\perp \cong \mathcal{C}_3/\mathcal{C}_4^\perp$. Note that, we do not necessarily have $\mathcal{C}_1/\mathcal{C}_2^\perp = \mathcal{C}_3/\mathcal{C}_4^\perp$, since $\mathcal{C}_2^\perp$ can be smaller than $\mathcal{C}_4^\perp$ in which case the cosets will have different cardinality. The two quotient groups are still isomorphic. □

If $\mathcal{C}_1 = \mathcal{C}_3$ and $\mathcal{C}_2 = \mathcal{C}_4$, the conditions of Theorem 1 are directly fulfilled. Thus, using the same CSS code at the stations results in CNOT transversality. This is in accordance with the fact that CSS-codes are CNOT transversal [25].

Note that using (21) and the CNOT transversality conditions (24), (25), we can rewrite the generators $\mathcal{C}_1$ and $\mathcal{C}_2$ allowing a transversal CNOT gate between stations A and B as

$$\mathbf{G}_1 = \begin{bmatrix}\mathbf{G}_2^\perp \\ \mathbf{A}\end{bmatrix}, \qquad \mathbf{G}_3 = \begin{bmatrix}\mathbf{G}_2^\perp \\ \mathbf{D} \\ \mathbf{A}\end{bmatrix}, \tag{29}$$

where $\mathbf{D}$ is the generator of $\mathcal{C}_4^\perp/\mathcal{C}_2^\perp$ which is a binary $(k_2 - k_4) \times n$ full rank matrix. It is worth to note that this structure implies that $\mathcal{C}_1 \subset \mathcal{C}_3$. Note that one could imagine from (29) that it is sufficient that $\mathcal{C}_1 \subset \mathcal{C}_3$ in order to have a transversal CNOT gate between the stations. This is, however, not correct. Consider, e.g., the case that $\mathcal{C}_1 = \mathcal{C}_4^\perp \subset \mathcal{C}_3$, i.e.,

$$\mathbf{G}_1 = \begin{bmatrix}\mathbf{G}_2^\perp \\ \mathbf{A}\end{bmatrix}, \qquad \mathbf{G}_3 = \begin{bmatrix}\mathbf{G}_2^\perp \\ \mathbf{A} \\ \mathbf{D}\end{bmatrix}.$$

In this example $\mathcal{C}_1 \subset \mathcal{C}_3$, however this configuration does not satisfy (25). In the following, we provide an example which has a transversal CNOT gate between two stations.

Example 1. *Transversal CNOT between two stations: Consider two CSS codes which map $k = 1$ logical qubits into $n = 7$ physical qubits: CSS$(\mathcal{C}_1, \mathcal{C}_2)$ with $k_1 = 4$, $k_2 = 4$ and CSS$(\mathcal{C}_3, \mathcal{C}_4)$ with $k_3 = 5$, $k_4 = 3$, having the following generators:*

$$\mathbf{G}_1 = \begin{bmatrix}\mathbf{G}_2^\perp \\ \mathbf{A}\end{bmatrix} = \left[\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hdashline 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\mathbf{G}_3 = \begin{bmatrix}\mathbf{G}_4^\perp \\ \mathbf{A}\end{bmatrix} = \begin{bmatrix}\mathbf{G}_2^\perp \\ \mathbf{D} \\ \mathbf{A}\end{bmatrix} = \left[\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hdashline 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ \hdashline 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

We then have a transversal non-local CNOT gate between stations A and B.

B. CZ Transversality

As a logical CNOT can be realized using a logical CZ across the two stations, and local operations at the stations, it is possible to base a 2G QR protocol based on logical CZ transversality. As we shall see, this poses different requirements on the structure of the codes. We have [32]:

Theorem 2. *Consider CSS$(\mathcal{C}_1, \mathcal{C}_2)$ and CSS$(\mathcal{C}_3, \mathcal{C}_4)$ codes in stations A and B, respectively, with generators given by* (21)*. We have a transversal CZ gate between station A and B iff*

$$\mathbf{x}^A \mathbf{z}^T + \mathbf{y}\left(\mathbf{x}^B + \mathbf{z}\right)^T = 0 \qquad \forall\, \mathbf{y} \in \mathcal{C}_2^\perp,\, \mathbf{z} \in \mathcal{C}_4^\perp \tag{30}$$

$$\mathbf{A}\mathbf{B}^T = \mathbf{I}, \tag{31}$$

Proof. Using a similar approach used for the transversality of CNOT gate, first note that by applying logical CZ gate on the logical states, we get

$$\mathbf{CZ}_{AB}^{L}|\boldsymbol{\psi}^{A}\rangle_{L}\otimes|\boldsymbol{\psi}^{B}\rangle_{L}=(-1)^{\boldsymbol{\psi}^{A}(\boldsymbol{\psi}^{B})^{T}}|\boldsymbol{\psi}^{A}\rangle_{L}\otimes|\boldsymbol{\psi}^{B}\rangle_{L} \quad (32)$$

If we apply CZ to the n pairs of physical qubits, we get the following state:

$$\begin{aligned}&\mathbf{CZ}_{AB}^{P}|\boldsymbol{\psi}^{A}\rangle_{L}\otimes|\boldsymbol{\psi}^{B}\rangle_{L}\\&=\alpha\sum_{\mathbf{y}\in\mathcal{C}_2^{\perp},\mathbf{z}\in\mathcal{C}_4^{\perp}}(-1)^{\left(\mathbf{x}^{A}+\mathbf{y}\right)\left(\mathbf{x}^{B}+\mathbf{z}\right)^{T}}|\mathbf{x}^{A}+\mathbf{y}\rangle\otimes|\mathbf{x}^{B}+\mathbf{z}\rangle\\&=\alpha\beta\sum_{\mathbf{y}\in\mathcal{C}_2^{\perp},\mathbf{z}\in\mathcal{C}_4^{\perp}}(-1)^{\mathbf{x}^{A}\mathbf{z}^{T}+\mathbf{y}\left(\mathbf{x}^{B}+\mathbf{z}\right)^{T}}|\mathbf{x}^{A}+\mathbf{y}\rangle\otimes|\mathbf{x}^{B}+\mathbf{z}\rangle\end{aligned} \quad (33)$$

where $\alpha=1/\sqrt{\left|\mathcal{C}_2^{\perp}\right|\left|\mathcal{C}_4^{\perp}\right|}$ and $\beta=(-1)^{\mathbf{x}^{A}\left(\mathbf{x}^{B}\right)^{T}}$. For having a transversal CZ gate, (32) should be equal to (33). According to the definition of the logical qubits, it is clear that (30) should be satisfied. Also, considering that $\mathbf{x}^{A}=\boldsymbol{\psi}^{A}\mathbf{A}$ and $\mathbf{x}^{B}=\boldsymbol{\psi}^{B}\mathbf{B}$, then $\beta=(-1)^{\boldsymbol{\psi}^{A}\mathbf{A}\mathbf{B}^{T}\left(\boldsymbol{\psi}^{B}\right)^{T}}$. Considering (32), we need to have $\beta=(-1)^{\boldsymbol{\psi}^{A}\left(\boldsymbol{\psi}^{B}\right)^{T}}$ which means that we have the $\mathbf{A}\mathbf{B}^{T}=\mathbf{I}$. This completes the proof. □

Corollary 1. *Consider CSS codes at stations A and B as in Theorem 2. A sufficient condition for having a transversal CZ gate between the stations is [32]*

$$\mathcal{C}_1/\mathcal{C}_2^{\perp}\cong\mathcal{A}_1,\quad\mathcal{C}_3\subseteq\mathcal{C}_2,\quad\mathbf{A}\mathbf{B}^{T}=\mathbf{I}, \quad (34)$$

or

$$\mathcal{C}_1\subseteq\mathcal{C}_4,\quad\mathcal{C}_3/\mathcal{C}_4^{\perp}\cong\mathcal{B}_1,\quad\mathbf{A}\mathbf{B}^{T}=\mathbf{I}, \quad (35)$$

where $\mathcal{A}_1\subset\mathcal{C}_4$ and $\mathcal{B}_1\subset\mathcal{C}_2$ are sets of vectors $\mathbf{c}\in\mathbb{F}_2^n$.

Proof. We note that for satisfying (30), one of the following conditions is sufficient

- $\mathbf{x}^{A}\mathbf{z}^{T}=0$ and $\mathbf{y}\left(\mathbf{x}^{B}+\mathbf{z}\right)^{T}=0$: this happens when $\mathcal{C}_1/\mathcal{C}_2^{\perp}\cong\mathcal{A}_1\subset\mathcal{C}_4$ and $\mathcal{C}_3\subset\mathcal{C}_2$.
- $\mathbf{x}^{A}\mathbf{z}^{T}=1$ and $\mathbf{y}\left(\mathbf{x}^{B}+\mathbf{z}\right)^{T}=1$: this cannot happen since $\mathbf{z}$ belongs to a linear code.
- $\left(\mathbf{x}^{A}+\mathbf{y}\right)\mathbf{z}^{T}=0$ and $\mathbf{y}\left(\mathbf{x}^{B}\right)^{T}=0$: this happens when $\mathcal{C}_1\subset\mathcal{C}_4$ and $\mathcal{C}_3/\mathcal{C}_4^{\perp}\subset\mathcal{C}_2$.
- $\left(\mathbf{x}^{A}+\mathbf{y}\right)\mathbf{z}^{T}=1$ and $\mathbf{y}\left(\mathbf{x}^{B}\right)^{T}=1$: this cannot happen since $\mathbf{z}$ belongs to a linear code.

Noting that $\mathbf{A}\mathbf{B}^{T}=\mathbf{I}$ should be satisfied as well completes the proof. □

C. Mirrored structure for CZ transversality

An example of CZ transversality between two stations is provided by *mirrored non-symmetric CCS codes*, defined as follows. Consider two classical linear codes, $\mathcal{C}_1$ and $\mathcal{C}_2$ with minimum distances d_1 and d_2, respectively. For constructing non-symmetric CSS code, CSS$\left(\mathcal{C}_1,\mathcal{C}_2\right)$, we assume that $\mathcal{C}_2^{\perp}\subset\mathcal{C}_1$ and $d_1>d_2$. We use $\mathcal{C}_1$ and $\mathcal{C}_2$ to correct Pauli $\mathbf{Z}$ and $\mathbf{X}$ errors, respectively.

We then construct another code CSS$(\mathcal{C}_3,\mathcal{C}_4)$ such that $\mathcal{C}_4^{\perp}\subset\mathcal{C}_3$ with $\mathcal{C}_3=\mathcal{C}_2$ and $\mathcal{C}_4=\mathcal{C}_1$. Therefore, in this code, $\mathcal{C}_2$ and $\mathcal{C}_1$ is used for correcting $\mathbf{Z}$ and $\mathbf{X}$ errors, respectively. The generator matrices of mirrored CSS codes can be written as:

$$\begin{bmatrix}\mathbf{G}_{\mathbf{2}}^{\perp}&\mathbf{0}\\\mathbf{0}&\mathbf{G}_{\mathbf{1}}^{\perp}\end{bmatrix}\text{ and }\begin{bmatrix}\mathbf{G}_{\mathbf{4}}^{\perp}&\mathbf{0}\\\mathbf{0}&\mathbf{G}_{\mathbf{3}}^{\perp}\end{bmatrix}=\begin{bmatrix}\mathbf{G}_{\mathbf{1}}^{\perp}&\mathbf{0}\\\mathbf{0}&\mathbf{G}_{\mathbf{2}}^{\perp}\end{bmatrix}, \quad (36)$$

where $\mathbf{0}$ is the all-zero matrix.

Example 2. *Consider a pair of mirrored CSS codes CSS$\left(\mathcal{C}_1,\mathcal{C}_2\right)$ and CSS$\left(\mathcal{C}_3,\mathcal{C}_4\right)$, between stations A and B with generator matrices [32]:*

$$\mathbf{G}_{\mathbf{1}}^{\perp}=\begin{bmatrix}1&1&0&0&1&0&0\\1&1&1&0&0&1&0\\1&1&1&0&0&0&1\end{bmatrix},$$

and

$$\mathbf{G}_{\mathbf{2}}^{\perp}=\begin{bmatrix}1&1&0&0&0&0&0\\0&1&0&1&1&1&1\end{bmatrix}.$$

These codes are CZ-transversal according to Corollary 1. After simple manipulations we find that

$$\mathbf{G}_4=\mathbf{G}_1=\begin{bmatrix}\mathbf{G}_{\mathbf{2}}^{\perp}\\\mathbf{A}\end{bmatrix}=\left[\begin{array}{ccccccc}1&1&0&0&0&0&0\\0&1&0&1&1&1&1\\\hline 0&0&1&1&0&1&1\\1&0&1&1&1&0&0\end{array}\right],$$

$$\mathbf{G}_2=\mathbf{G}_3=\begin{bmatrix}\mathbf{G}_{\mathbf{4}}^{\perp}\\\mathbf{B}\end{bmatrix}=\left[\begin{array}{ccccccc}0&0&0&0&1&1&0\\0&0&0&0&1&0&1\\1&1&0&0&0&1&0\\\hline 0&1&1&1&0&0&1\\0&1&1&0&0&0&1\end{array}\right].$$

Now $\mathcal{C}_1/\mathcal{C}_2^{\perp}\subset\mathcal{C}_4$, $\mathcal{C}_3=\mathcal{C}_2$, and $\mathbf{A}\mathbf{B}^{T}=I_2$. Thus the conditions (34) are satisfied and this pair of mirrored CSS codes is indeed CZ-transversal.

It is straight forward to see that any CSS codes with generators (36), satisfy the first two conditions of (34). By analyzing the third condition we have:

Corollary 2. *Consider a pair of mirrored CSS codes with $\mathcal{C}_1=\mathcal{C}_4$ and $\mathcal{C}_2=\mathcal{C}_3$. Any such pair can be transformed to satisfy the sufficient conditions for CZ transversality by row-wise operations on matrix $\mathbf{A}$ or $\mathbf{B}$.*

Proof. In the mirrored structure the component code generators of (21) can be written as

$$\mathbf{G}_1=\begin{bmatrix}\mathbf{G}_3^{\perp}\\\mathbf{A}\end{bmatrix},\qquad\mathbf{G}_3=\begin{bmatrix}\mathbf{G}_1^{\perp}\\\mathbf{B}\end{bmatrix}. \quad (37)$$

From the mirrored structure it directly follows that the first condition of (34) is fulfilled, as $\mathcal{C}_1/\mathcal{C}_2^{\perp}\subset\mathcal{C}_1=\mathcal{C}_4$, while the second condition is fulfilled by definition. To prove that mirrored structure codes are CZ transversal, we need to show that $\mathbf{A}\mathbf{B}^{T}=\mathbf{I}$. In general rank$\left(\mathbf{A}\mathbf{B}^{T}\right)\leq\min\left(\text{rank}\left(\mathbf{A}\right),\text{rank}\left(\mathbf{B}\right)\right)$. However, due to the structure in (37), $\mathbf{A}\mathbf{B}^{T}$ is a full rank matrix.

To prove this, assume that it is not full rank. The (i,j)-th entry of $\mathbf{AB}^T$ is $\mathbf{a}_i\mathbf{b}_j^T$, where $\mathbf{a}_i$ and $\mathbf{b}_j$ are ith and jth row of $\mathbf{A}$ and $\mathbf{B}$, respectively. If $\mathbf{AB}^T$ is not full rank, its columns are dependent; there exists a binary vector $\mathbf{x}=(x_1,x_2,...,x_n)$ such that $\sum_i x_i\,\mathbf{a}_i\mathbf{B}^T=\mathbf{0}$. If we define $\mathbf{a}'=\Sigma_i x_i\mathbf{a}_i$, we then have $\mathbf{a}'\in\mathbf{B}^\perp$. According to (36), for mirrored CSS codes, code spaces and their duals are related as

$$\mathbf{G_1}=\begin{bmatrix}\mathbf{G_3^\perp}\\ \mathbf{A}\end{bmatrix},\quad \mathbf{G_3}=\begin{bmatrix}\mathbf{G_1^\perp}\\ \mathbf{B}\end{bmatrix}.$$

For a rank-deficient $\mathbf{AB}^T$ we thus would have $\mathbf{G}_3^\perp=\mathbf{G}_1\cap\mathbf{B}^\perp$, as $\mathbf{a}'$ is a linear combination of $\mathbf{a}_i\in\mathbf{A}\subset\mathbf{G_1}$. Thus $\mathbf{a}'\in\mathbf{G}_3^\perp$, which means that $\mathbf{G}_1$ is not full rank, which is a contradiction. Thus $\mathbf{AB}^T$ has to be full rank. From this it follows that using Gaussian elimination we can find a $\mathbf{W}$ such that $\mathbf{WAB}^T=\mathbf{A}'\mathbf{B}^T=\mathbf{I}$. This means that by selecting a proper $\mathbf{A}$ in station A, CZ transversality is achievable. □

D. Hadamard gate transversality

In second generation QRs, we need logical CNOT gates to establish logical entanglement. A CZ gate can be converted to a CNOT gate with two additional Hadamard gates.

In the literature [25], the Hadamard gate transversality is proved via stabilizer generator algebra. For a $\mathrm{CSS}(\mathcal{C}_1,\mathcal{C}_2)$ code with generators given by (21), the code has a transversal Hadamard gate iff $\mathcal{C}_1=\mathcal{C}_2$. Here we verify Hadamard gate transversality in our framework, in order to find out whether the encoding matrix has an effect on Hadamard gate transversality.

The action of m parallel Hadamard gates on m-qubit state $|\mathbf{a}\rangle$ can be expressed as:

$$\mathbf{H}^{\otimes m}|\mathbf{a}\rangle=\frac{1}{\sqrt{2^m}}\sum_{\mathbf{b}\in\mathbb{F}_2^m}(-1)^{\mathbf{ab}^T}|\mathbf{b}\rangle. \tag{38}$$

For $[[n,k,d]]$ CSS code with the generator matrix as (3), it has Hadamard gate transversality, when it satisfies $\mathbf{H}_L^{\otimes k}|\boldsymbol{\psi}\rangle_L=\mathbf{H}^{\otimes n}|\boldsymbol{\psi}\rangle$, where $\mathbf{H}_L$ indicates logical Hadamard gate, and $|\boldsymbol{\psi}\rangle$ denote the corresponding n qubits physical state. We can write the logical parallel Hadamard gate acting on the logical state as:

$$\begin{aligned}\mathbf{H}_L^k|\boldsymbol{\psi}\rangle_L&=\frac{1}{\sqrt{2^k}}\sum_{\boldsymbol{\psi}_b\in\mathbb{F}_2^k}(-1)^{\boldsymbol{\psi}\boldsymbol{\psi}_b^T}|\boldsymbol{\psi}_b\rangle_L\\&=\frac{1}{\sqrt{2^k}\sqrt{|\mathcal{C}_2^\perp|}}\sum_{\boldsymbol{\psi}_b\in\mathbb{F}_2^k}\sum_{\mathbf{y}\in\mathcal{C}_\mathbf{2}^\perp}(-1)^{\boldsymbol{\psi}\boldsymbol{\psi}_b^T}|\mathbf{A}\boldsymbol{\psi}_b+\mathbf{y}\rangle,\end{aligned} \tag{39}$$

where $\mathbf{A}$ is the mapping matrix in (21). Since $\mathbf{A}\in\mathcal{C}_1/\mathcal{C}_2^\perp$ and $\boldsymbol{\psi}_b$ is summed over $\mathbb{F}_2^k$, the double summation over $\boldsymbol{\psi}_b$ and $\mathbf{y}$ consists all the codewords in $\mathcal{C}_1$.

We also can write n parallel physical Hadamard gates acting on the logical state as:

$$\begin{aligned}\mathbf{H}^{\otimes n}|\boldsymbol{\psi}\rangle_L&=\frac{1}{\sqrt{|\mathcal{C}_2^\perp|}}\sum_{\mathbf{z}\in\mathcal{C}_2^\perp}\mathbf{H}^{\otimes n}|\mathbf{x}+\mathbf{z}\rangle\\&=\frac{1}{\sqrt{2^n}\sqrt{|\mathcal{C}_2^\perp|}}\sum_{\mathbf{z}\in\mathcal{C}_2^\perp}\sum_{\mathbf{x}'_b\in\mathbb{F}_2^n}(-1)^{(\mathbf{x}+\mathbf{z})\mathbf{x}_b'^T}|\mathbf{x}'_b\rangle\\&=\frac{\sqrt{|\mathcal{C}_2^\perp|}}{\sqrt{2^n}}\sum_{\mathbf{x}'_b\in\mathcal{C}_2}(-1)^{\boldsymbol{\psi}\mathbf{A}\mathbf{x}_b'^T}|\mathbf{x}'_b\rangle.\end{aligned} \tag{40}$$

If we want (39) to equal (40), it is a necessary condition that $\mathcal{C}_2=\mathcal{C}_1$, then these two equations contain the same physical states.

Then we can check the normalize parameter of two equations, since we assume $[[n,k,d]]$ CSS code with $\mathcal{C}_2=\mathcal{C}_1$, we can get the classical code has parameter $[n,k']$ as $2k'-n=k$. The $\mathcal{C}_2^\perp$ has cardinality as $|\mathcal{C}_2^\perp|=2^{\frac{n-k}{2}}$. We can rewrite the parameter of (40) as:

$$\frac{\sqrt{|\mathcal{C}_2^\perp|}}{\sqrt{2^n}}=\sqrt{2}^{\frac{n-k}{2}-n}=\frac{1}{\sqrt{2^{\frac{n+k}{2}}}}=\frac{1}{\sqrt{|\mathcal{C}_2^\perp|}\sqrt{2^k}},$$

we get the same normalize parameter for both equations.

The last thing for equality to hold, the sign for each state should coincide, up to a possible global phase. To address the same physical state in both equations, we have $\mathbf{x}'_b=\mathbf{A}\boldsymbol{\psi}_b+\mathbf{y}$. We can rewrite the phase factor in (40) as

$$(-1)^{\boldsymbol{\psi}\mathbf{A}\mathbf{x}_b'^T}=(-1)^{\boldsymbol{\psi}\mathbf{A}\mathbf{A}^T\boldsymbol{\psi}_b^T+\boldsymbol{\psi}\mathbf{A}\mathbf{y}^T}.$$

Since $\boldsymbol{\psi}\mathbf{A}\in\mathcal{C}_1$, $\mathbf{y}\in\mathcal{C}_2^\perp$ and $\mathcal{C}_1=\mathcal{C}_2$, we have $\boldsymbol{\psi}\mathbf{A}\mathbf{y}^T=\mathbf{0}$. The only possibility for (39) to equal (40) up to a global phase thus is $\mathbf{AA}^T=\mathbf{I}$. Note that while this seems an additional condition for Hadamard transversality, it can be readily understood from the role of the mapping matrix $\mathbf{A}$. We shall see in Sec. IV-E that the rows of $\mathbf{A}$ represent logical $\mathbf{X}$ operators of the quantum code. The condition $\mathbf{AA}^T=\mathbf{I}$ thus naturally holds for symmetric CSS codes as a consequence of the commutation rules of logical operators.

For use in 2G QRs, to create a non-local CNOT gate we can combine a transversal Hadamard gate with a transversal non-local CZ gate to achieve a transversal CNOT gate in a specific equation. Recall that for CZ transversality, the codes need to satisfy (34) or (35), as well as the transversal Hadamard gate condition $\mathcal{C}_1=\mathcal{C}_2$.

Example 3. *Consider a pair of CZ transversal CSS codes* $\mathrm{CSS}(\mathcal{C}_1,\mathcal{C}_2)$ *and* $\mathrm{CSS}(\mathcal{C}_3,\mathcal{C}_4)$*, with* $\mathrm{CSS}(\mathcal{C}_1,\mathcal{C}_2)$ *also has Hadamard transversality, between station A and B with generator matrices:*

$$\mathbf{G}_1=\begin{bmatrix}\mathbf{G}_2^\perp\\ \mathbf{A}\end{bmatrix}=\begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\\ \hline 1&1&1&0&0&0&0\end{bmatrix}.$$

$\mathrm{CSS}(\mathcal{C}_1, \mathcal{C}_2)$ *is Steane code with transversal Hadamard gate. According to transversal CZ gate conditions* (34)*, we can find the paired code* $CSS(\mathcal{C}_3, \mathcal{C}_4)$ *as:*

$$\mathbf{G}_3^{\perp} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and

$$\mathbf{G}_4 = \begin{bmatrix} \mathbf{G}_3^{\perp} \\ \mathbf{A} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have $\mathcal{C}_1/\mathcal{C}_2^{\perp} \cong \mathcal{A}$ *and* $\mathcal{A} \subset \mathcal{C}_4$*. Since* $\mathcal{C}_2^{\perp} \subset \mathcal{C}_3^{\perp}$*, we have* $\mathcal{C}_3 \subset \mathcal{C}_2$ *which fulfills the condition* (34)*.*

In the case of having both CZ and Hadamard gate transversality, one of the two codes used in neighboring stations is a symmetric CSS code, which means that it is not designed for the biased error model (18) governing the 2G QR quantum channel. Thus, for $\mathcal{Q}_{12}$ and $\mathcal{Q}_{34}$ used in station A and B, choosing $\mathcal{Q}_{12}$ as a symmetric and $\mathcal{Q}_{34}$ as an asymmetric code will lead to worse performance than designing $\mathcal{Q}_{12}$ and $\mathcal{Q}_{34}$ according to the mirrored structure, if the codes have similar minimum distance. The latter construction would, however, necessitate non-transversal Hadamard gates.

There exists a low-cost fault-tolerant manner to realize the logical Hadamard gate for small QECCs [33]. Importantly this method does not need extra ancilla qubits. As a consequence, we consider mirrored structure CZ-transversal code for 2G QR in the simulations, and neglect the cost of local logical Hadamard gates. Keeping only CZ transversality between neighboring stations performs better than also adding a transversal Hadamard restriction on one end.

E. Relation of Encoding and Transversality

As we showed in the previous sections the matrices $\mathbf{A}$ and $\mathbf{B}$ introduced in (21) may affect the transversality of the corresponding CSS codes.

It is worth noting that $\mathbf{A}$ and $\mathbf{B}$ define the encoding mapping from logical qubits to physical qubits. Let $\psi_i^A = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the logical state with 1 in the i-th position. Then, according to (22), we have

$$|\boldsymbol{\psi}_i^A\rangle = \frac{1}{\sqrt{\mathcal{C}_2^{\perp}}} \sum_{\mathbf{y} \in \mathcal{C}_2^{\perp}} |\mathbf{a}_i + \mathbf{y}\rangle.$$

For the same reason we have

$$|0, 0, ..., 0\rangle_L = \frac{1}{\sqrt{\mathcal{C}_2^{\perp}}} \sum_{\mathbf{y} \in \mathcal{C}_2^{\perp}} |\mathbf{y}\rangle.$$

If we denote by $\mathbf{X}_L^i \in \mathbb{C}^{2^k}$ the logical Pauli gate that acts by $\mathbf{X}$ on logical qubit i, and does not rotate other qubits, then from the above two equations we have

$$\mathbf{X}_L^i |0, 0, ..., 0\rangle_L = \frac{1}{\sqrt{\mathcal{C}_2^{\perp}}} \sum_{\mathbf{y} \in \mathcal{C}_2^{\perp}} |\mathbf{a}_i + \mathbf{y}\rangle. \tag{41}$$

This means that the action of $\mathbf{X}_L^i$ corresponds to adding $\mathbf{a}_i$ to the physical qubits.

From Theorem 1 we know that in order for CSS codes used in station A and station B to be CNOT-transversal it is necessary that $\mathbf{A} = \mathbf{B}$. Combining this with (41), we conclude that for CNOT-transversality it is necessary that realizations of logical $\mathbf{X}$ gates for these codes are identical, despite that the codes themselves are different. Equivalently, the encoding in stations A and B should be the same.

Let us now consider logical CZ-transversality. First, we would like to recall that it was shown in [12], [13] that any self-orthogonal CSS code $\mathrm{CSS}(\mathcal{C}, \mathcal{C})$ is CZ-transversal. This means that one can use any encoding and still have CZ-transversality between any two code vectors. However, from Theorem 1 and Corollary 1 it follows that this is not the case for code vectors from two different CSS codes, which could be used, for example, in stations A and B. Indeed, in this case the encodings must be such that they guarantee the property $\mathbf{A}\mathbf{B}^T = \mathbf{I}$.

It is worth to note that self-orthogonal CSS codes $\mathrm{CSS}(\mathcal{C}, \mathcal{C})$ are a special case of the mirrored structure defined in (36). Thus, our proof of CZ-transversality of the mirrored CSS codes gives an alternative proof that self-orthogonal CSS codes are CZ-transversal.

To conclude this section, we would like to point out that using the same self-orthogonal CSS code in stations A and B will guarantee CNOT- and CZ-transversality despite the choice of encoding operations in station A and B. It is also efficient for correcting depolarizing errors. However, when the error model is structured, as in the 2G QR system, using a pair of nonidentical CZ-transversal CSS codes specifically designed for the error model will perform better. In Section VI we shall verify by numerical simulations, that using a pair of nonidentical CZ-transversal CSS codes in neighboring stations provides an order of magnitude improvement in the fidelity compared to the case of using the same codes.

F. Universality and transversality

In this paper, for use in 2G QR, we study the transversality conditions between different codes, which seems like a potential way to bypass the Eastin-Knill theorem [34]. Here we will show it is not possible to achieve universal transversal computation with the $\{\mathbf{CNOT}\,(\text{or}\,\mathbf{CZ}), \mathbf{T}, \mathbf{H}\}$ gate set even when different codes are used to protect different logical bits. Conditions on CSS codes to have a transversal $\mathbf{T}$ gate have been addressed in the literature [35].

For universal computation with CNOT gates, CNOT gates acting in both directions are needed. Conditions (24) and (25) only guarantee CNOT transversality gate from station A to B. By doubling the condition for both directions, we find that two CSS codes $\mathrm{CSS}(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathrm{CSS}(\mathcal{C}_3, \mathcal{C}_4)$ have transversal CNOT gates in both directions only when $\mathcal{C}_1 = \mathcal{C}_3$ and $\mathcal{C}_2 = \mathcal{C}_4$, which means that they are the same code. In such a case, universality and transversality will be limited by Eastin-Knill theorem.

Now instead of CNOT gates, consider using CZ gate to achieve universality. Since the CZ gate does not have control or target qubits, we do not need to consider two directions. Having transversal CZ across two codes, with transversal $\mathbf{T}$ gate on one end and $\mathbf{H}$ at the other is not sufficient for universality, since the CZ gate commutes with the $\mathbf{T}$ gate, the non-Clifford gates on one side of CZ gate can not propagate to another end. Thus, the codes on both ends need to support both transversal $\mathbf{T}$ and Hadamard gates, which conflicts with Eastin-Knill theorem.

V. Rate of Quantum Communications over 2G QRs

To quantify the performance of a quantum repeater network, we can use two parameters. In addition to logical entanglement fidelity, EGR, indicating the rate of generating end-to-end entangled qubits, is an important factor. These characterize performance irrespectively of whether entanglement is used to generate cryptographic keys, or for classical or quantum communication.

Here, we analyze logical entanglement generation rate for 2G QRs in a system where the overall distance L_T is divided to S hops. The stations are separated with the same nearest neighbor distance $L_0 = L_T/S$. We use M levels for the Bell pair purification protocol, and an $[[n,k]]$ QECC. For the situation of finite-size quantum memory, we use m to indicate the number of qubits in quantum memory.

In the 2G QRs system, end-to-end logical entanglement is established in two steps. First, we generate logical entanglement between neighboring stations. Second, by using entanglement swapping we can achieve overall entanglement. As logical entanglement is generated by transmission of physical Bell pairs, the EGR of 2G QRs is limited by the speed of generating physical Bell pairs. We define EGR as:

$$R = \frac{1}{\tau} \tag{42}$$

where τ is the time it takes to generate one pair of end-to-end entangled logical qubits. It contains three parts:

$$\tau = \frac{N_r}{R_{raw}} + t_{op} + t_{tr} \tag{43}$$

where R_{raw} is the raw Bell pair generation rate, N_r the number of raw Bell pairs consumed to generate one purified pair, t_{op} the operation time for all local operations, and $t_{tr} = L_0/c$ is the transmission duration, with c the speed of light. Assuming that local operations happen in a nanosecond time scale [29], t_{op} is negligible and can be omitted.

There are two processes that consume raw Bell pairs during purification: erasure during transmission, and purification failures. The erasure error rate during transmission is e_0 of (5). The erasure error rate is directly related to the neighboring distance L_0.

Purification success can be summarized as an overall purification success probability p_{tot}. It is related to local operation and physical Bell pair fidelity. In round i of purification, we have success probability p_i of (12). We thus have to consume $2/p_i$ pairs of the previous level for one successfully purified pair on level i. Thus for one pair on level M, we need to start with $2^M/p_{tot}$ raw Bell pairs, with $p_{tot} = \prod_{i=1}^{M} p_s^i$.

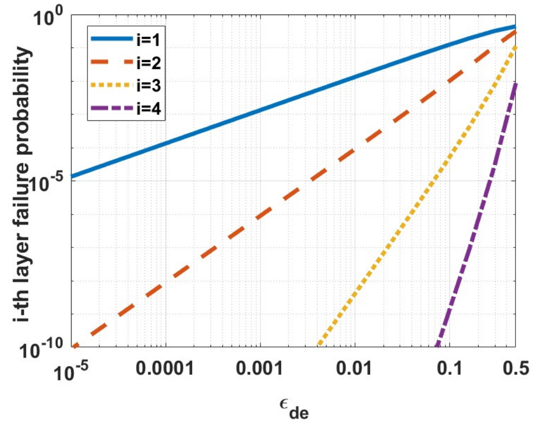

Fig. 2. The initial depolarizing error rate v.s ith layer purification failure probability.

Taking erasures into account, for M-level purification, we thus need $2^M/p_{tot}(1-e_0)$ raw Bell pairs to successfully generate one purified Bell pair. Furthermore, for each logical qubit, we need n/k purified Bell pairs. The number of consumed Bell pairs N_r thus becomes:

$$N_r = \frac{2^M n}{(1-e_0)\, p_{tot}\, k}\,. \tag{44}$$

The relative size of the terms in (43) varies according to the hardware used. In general, if the neighboring distance L_0 is more than 10 km, the qubit generation time cost N_r/R_{raw} is much smaller than the transmission duration t_{tr} [29].

To model errors in Bell pair purification we assume that the raw Bell pairs suffer from depolarizing errors, such that $f_1 = f_2 = f_3 = \varepsilon_{de}/3$. The purification success probabilities (12) can then be solved from the recursion (11). In Fig.2, we have plotted p_s^i as a function of ε_{de} for $i = 1, \ldots, 4$. For all values $\varepsilon_{de} < 0.5$, the second level purification failure probability is much smaller than the first level; $(1 - p_s^i) \ll (1 - p_s^1)$. Thus we can ignore the purification success failures for levels $i > 1$, and assume $p_{tot} \approx p_1$.

The time consumption of the transmission procedure depends on the size of the quantum memory. With unlimited quantum memory, an unlimited number of Bell-pairs can be transmitted, and all the qubits needed for purification can be stored in quantum memory to wait for further operations. Thus, the procedure suffers from the delay only in the setup phase from the first transmission, and the following classical communication for purification measurement. Under these circumstances, only the distance between neighboring stations L_0 would affect the transmission time cost.

The situation changes if there is limited quantum memory. In order to finish purification, several rounds of transmission are needed. The transmission time depends on the required number of Bell pairs N_r, and will grow along with the selected purification level M and purification success probability p_s. The time cost of transmission procedure comes from the following steps:

1) Transmitter sends m photonic qubits through the fiber to the receiver. We assume m is smaller than the number N_r of required qubits from (44).
2) The receiver uses heralded entanglement generation to correct erasure errors with the help of classical communication. The delay of this procedure is taken into account in the e_0-dependence of N_r.
3) Transmitter sends additional qubits to fill up the quantum memory. On average, $\frac{N_r}{m}$ rounds of transmission will be needed. For each round, there are two communication events, first the qubit transmissions, then the classical feedback. The average time cost thus is $2\frac{N_r}{m}\frac{L_0}{c}$.

From this, we get the EGR for a limited quantum memory situation:

$$\tau = \frac{N_r}{m}\left(\frac{m}{R_{raw}} + 2\frac{L_0}{c}\right) + M\frac{L_0}{c} \tag{45}$$

Here the factor 2 in the second term in the parenthesis indicates two-way, quantum and classical communication with heralded entanglement generation. The last term is contributed by classical commination in the M-level purification procedure.

In a practical scenario, we consider a system built for generating end-to-end entanglement between two stations far apart. The quantum communication service requirements are expressed in terms of a EGR and overall logical fidelity thresholds. The overall logical fidelity can be expressed as

$$f_T = f_L^{2S}, \tag{46}$$

where f_L is logical fidelity in each code block and S is the number of hops. The logical fidelity f_L is a function of the neighboring distance L_0, purification level M, and local gate error rate f_g. The cost of building the system is computed as the number of needed intermediate stations. To minimize cost, the neighboring distance L_0 should be maximized. This leads to the optimization problem:

$$\begin{aligned} & \max L_0 \\ s.t \quad & R(L_0, M) \geq R_{\text{th}} \\ & f_T\left(L_0, M, f_g\right) \geq f_{\text{th}}\,. \end{aligned}$$

Here R_{th} and f_{th} are desired thresholds for EGR and the overall logical fidelity for reliable quantum communication. With fixed overall distance L_T, by maximizing L_0, we get the minimum number of intermediate stations. We shall consider this optimization numerically in the next section.

VI. Numerical Results

In this section, we provide numerical results to evaluate the infidelity and EGR in quantum repeaters. We use the distance-dependent error model (7) with $\alpha = 0.17$ dB/km and $\beta = 0.01$ /km. With neighboring distance L_0, the raw Bell pairs (9) after transmission have $f_1 = f_2 = f_3 = \left(1 - e^{-0.01L_0}\right)/3$ [36] as initial values of the recursion (11). For infidelity simulation, we consider different levels M of purification, and use the maximum likelihood decoder [37]. For numerical EGR results, we set the photonic qubit generation rate as $R_{raw} = 10^6$ [29].

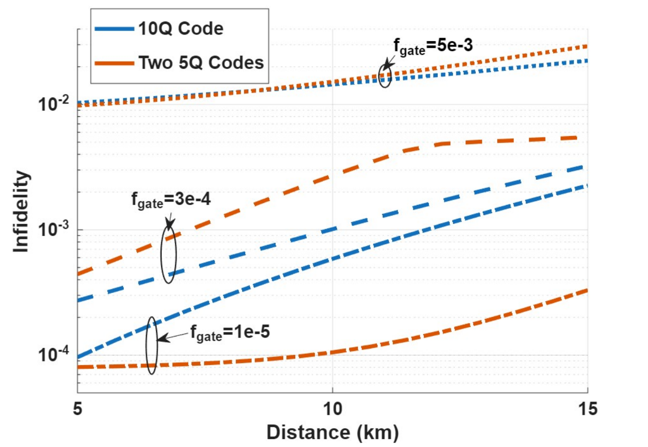

Fig. 3. Performance in correlated error channel (18). Cross-over behavior between symmetric (10-qubit) and non-symmetric (two 5-qubit) codes with same minimum distance. The infidelity as a function of distance between neighboring stations for different two-qubit gate error rates.

We neglect photonic qubit detection errors and quantum memory decoherence. For limited quantum memory situation, we assume that each station has $m = 100$ memory qubits for storing information.

First, following Section III-E, we compare symmetric and non-symmetric codes with the same minimum distance, which means they have similar performance for depolarizing errors. We study their infidelity performance under the correlated error model (18). In Fig. 3, we compare the performance of two alternatives, one is using a symmetric $[[10, 2, 3]]$ CSS code to protect two logical qubits at each station, another one is using two 5-qubit non-symmetric codes in parallel as $[[5, 1, 3]]^{\otimes 2}$ codes to protect the two logical qubits, in channel (18). Note that here, the same code is used at both station A and B, and joint decoding is performed between neighboring stations using a maximum-likelihood algorithm. The purification level is $M = 4$. The infidelity of the logical qubits is plotted against gate error probability and distance.

In the large neighboring distance area, where the correlated errors are dominant, the non-symmetric codes achieve better performance. This is due to the fact that the non-symmetric codes are designed to handle correlated **ZX** errors. This is concordance with Proposition 1, which states that a non-symmetric CSS code outperforms a symmetric code under correlated error scenario. When depolarizing gate errors dominate, these two codes have similar performance since they have the same minimum distance.

Next, following Section IV, we consider using pairs of different codes at neighboring stations under error model (16) and purification level $M = 4$. We compare the performance of pairs of codes which are CNOT or CZ transversal with the case of using the same CSS code at neighboring stations. In this case, joint decoding is unnecessary and maximum-likelihood decoding is used separately at each station instead.

Fig. 4 compares the infidelity with the CNOT transversal structures. In this figure, $\mathcal{Q}_{12} = \text{CSS}\left(\mathcal{C}_1, \mathcal{C}_2\right)$ and $\mathcal{Q}_{34} =$

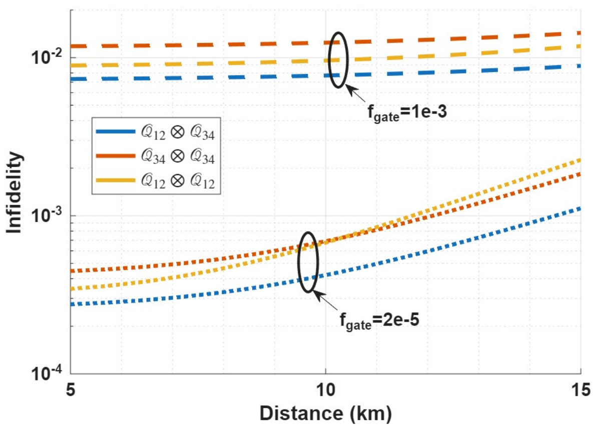

Fig. 4. The infidelity as a function of distance between neighboring stations with CNOT transversal structure, for different two-qubit gate error rates. Codes from Example 1.

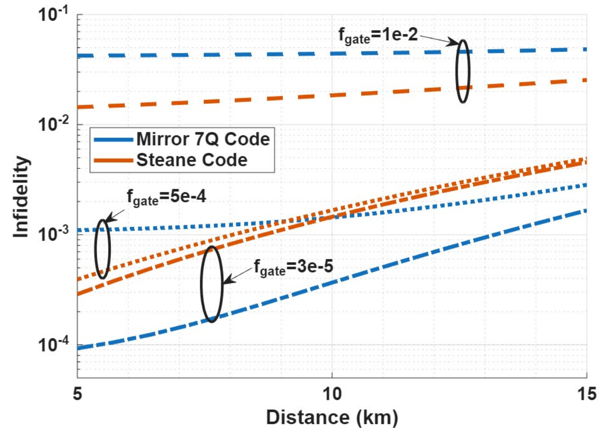

Fig. 5. Comparison between $[[7,1,3]]$ Steane code (CNOT transversal) and $[[7,1,2]]$ non-symmetric code with mirrored structure (CZ transversal).

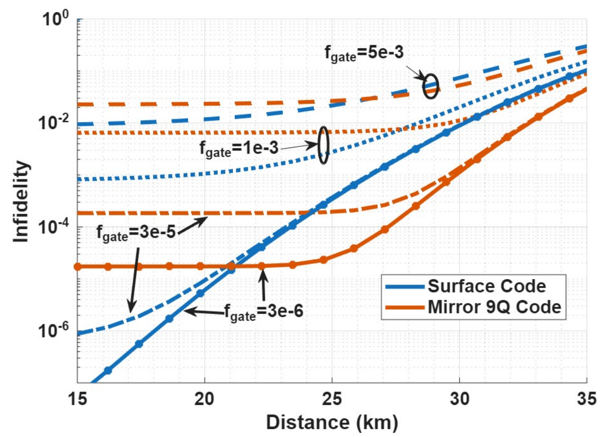

Fig. 6. Comparison between $[[9,1,3]]$ distance-3 surface code (CNOT transversal) and $[[9,1,2]]$ non-symmetric code with mirrored structure (CZ transversal).

$\mathrm{CSS}\left(\mathcal{C}_3, \mathcal{C}_4\right)$ are CSS codes $[[7,1,2]]$ based on Example 1. Note that the minimum distances of $\mathcal{Q}_{12}$ and $\mathcal{Q}_{34}$ are $d=2$. In this figure $\mathcal{Q}_{12} \otimes \mathcal{Q}_{34}$ means that station A and B are using $\mathcal{Q}_{12}$ and $\mathcal{Q}_{34}$, respectively. $\mathcal{Q}_{12}$ and $\mathcal{Q}_{34}$ are designed to satisfy the CNOT transversality condition. It is seen that the use of the $\mathcal{Q}_{12} \otimes \mathcal{Q}_{34}$ CSS structure results in better performance in comparison to using the same code at the stations.

In Fig. 5, we also compare the logical infidelity between $[[7,1,3]]$ symmetric Steane code and $[[7,1,2]]$ mirrored structure non-symmetric code, which consist of $\mathcal{C}_1\left(7,2,4\right)$ with generator matrix

$$\mathbf{G}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and $\mathcal{C}_2\left(7,6,2\right)$ with the parity check matrix

$$\mathbf{G}_2^{\perp} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

at station A, and has 5 Pauli-X stabilizers and 1 Pauli-Z generators. In station B, we use code $\mathrm{CSS}\left(\mathcal{C}_2, \mathcal{C}_1\right)$. When the biased error is significant, i.e., when the neighboring distance L_0 is relatively large, the mirrored structure outperforms the surface code, despite its smaller minimum distance.

In Fig. 6, we compare the performance of the mirrored approach with the case of using similar symmetric CSS codes in both stations of a QR system. We consider purification level $M=6$. For mirrored structure, in station A, $\mathcal{C}_1\left(9,2,5\right)$ with generator

$$\mathbf{G}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and $\mathcal{C}_2$ $\left(9,8,2\right)$ with the parity check matrix

$$\mathbf{G}_2^{\perp} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

are used to construct a non-symmetric $[[9,1,2]]$ CSS code, which has 7 Pauli-X stabilizers and 1 Pauli-Z generators. In station B, we use code $\mathrm{CSS}(\mathcal{C}_3, \mathcal{C}_4)$ where $\mathcal{C}_3 = \mathcal{C}_2$ and $\mathcal{C}_4 = \mathcal{C}_1$. As a result, this quantum code has one Pauli-X stabilizer and 7 Pauli-Z generators. With the mirrored approach we have CZ transversality in neighboring stations. This is benchmarked against the $[[9,1,3]]$ surface code, which is symmetric, has CNOT transversality and has distance-3.

In Fig. 7, we compare the infidelity of several codes: Steane code, distance-3 surface code, CNOT transversal codes $\mathcal{Q}_{12} \otimes \mathcal{Q}_{34}$, and the mirrored structure 9-qubit and 7-qubit codes. The gate error rate is set to 10^{-5}. In the regime of large neighboring distances, where the biased errors dominate, the mirrored structure codes have lower infidelity than others, as they are specifically designed to handle such errors. The 9-qubit mirrored code outperforms the 7-qubit code in the mid-distance range (from 20 km to 35 km). However, as the distance increases further, biased errors accumulate significantly across the larger code block, such that the infidelity of the 9-qubit mirrored code approaches the 7-qubit one. In such cases, achieving better performance with larger codes would require increasing the purification level to better suppress biased errors.

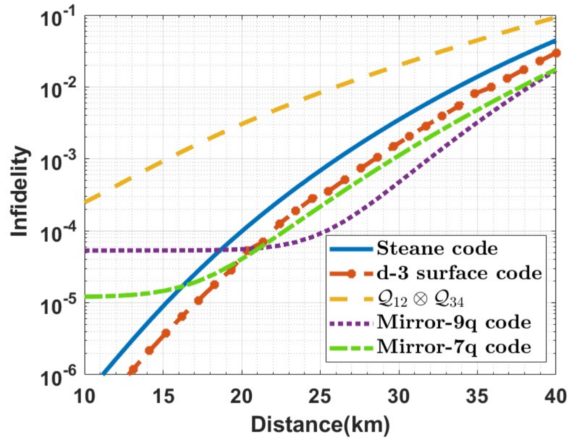

Fig. 7. Relation between infidelity of different codes and neighboring station distance. The local gate error rate is set to 10^{-5}.

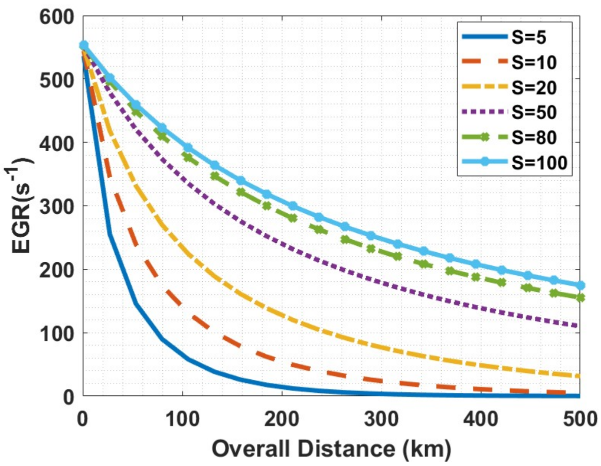

Fig. 8. Relation between EGR and overall distance, for different numbers S of intermediate stations. We choose 9-qubit mirrored structure code, set the purification level $M = 6$ and local operation error $f_g = 10^{-5}$.

At lower distances, where gate errors dominate, conventional codes outperform mirrored structure due to their larger code distance. In this regime, the 7-qubit mirrored code achieves higher fidelity than the 9-qubit version, as it involves fewer gate operations. It is also worth noting that the CNOT transversal code always has the highest infidelity among all codes considered, due to its lower code distance and not designed for protecting against biased errors.

Next, we analyze the effect of the code structure and channel parameters on communication performance. First, in Fig. 8, we show the relation between EGR and the overall distance for different numbers of intermediate stations with 9-qubit codes used in Fig. 6. Note that using the same distance-3 surface code or the mirrored structure non-symmetric codes do not affect the EGR of (45), it only affects the infidelity. When the overall distance is small, the EGR approaches an upper bound given by the photonic qubit generation rate.

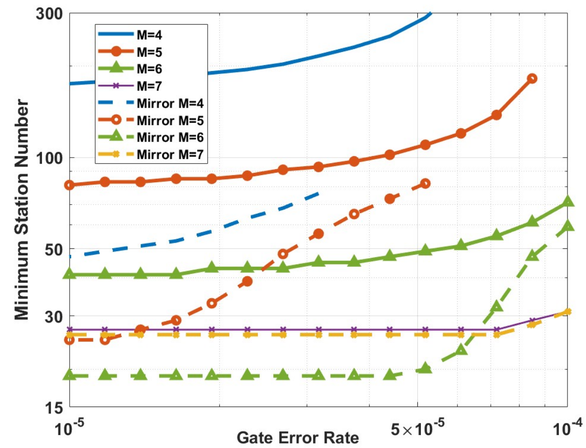

Fig. 9. Minimum intermediate station number vs local gate error rate, for overall distance 500 km, EGR $R_{\mathrm{th}} = 50$ entangled qubits/s and overall logical fidelity threshold $f_{\mathrm{th}} = 0.95$. The parameter M denotes the purification level of the Bell states. Steane code compared to mirrored CSS 7-qubit code with purification level $M = 4, 5, 6, 7$.

In Fig. 9, we compare the minimum number of intermediate stations when using 7-qubit mirrored structure $[[7, 1, 2]]$ code and the $[[7, 1, 3]]$ Steane code, as in the Fig. 5, given EGR and fidelity threshold requirements, $R_{\mathrm{th}} = 50$ qubit$/s$ and $f_{\mathrm{th}} = 0.95$. When the gate error approaches 10^{-5}, the distance-related errors dominate, and fulfilling the logical fidelity threshold becomes independent of the local gate error rate. For purification levels $M = 4, 5, 6$, using mirrored structure codes can reduce the number of stations significantly. This is because the mirrored structure codes can correct more distance-related errors, thus have much higher logical fidelity. However, for $M = 4, 5$ and a large gate error rate, we cannot find a station number for the mirrored structure that meets all the thresholds. When we set $M = 7$, using both codes can have high logical fidelity above the fidelity threshold. In this situation, the minimum station number is limited by EGR. Since both codes use the same number of qubits, their EGRs are the same. Thus, with larger purification level, codes have higher fidelity, but the EGR is limited by the larger resource cost. Overall, in this tradeoff, the smallest number of stations is provided by $M = 6$ and the mirrored structure for gate error rate $< 5 \times 10^{-5}$.

Finally, in Fig. 10, we compare between larger codes, the 9-qubit mirrored structure codes and distance-3 surface code, as in the Fig. 6, with the same parameters as in Fig. 9. Comparing with 7-qubit codes, we can see that 9-qubit codes need less intermediate stations in the low gate error rate region. The 9-qubit code reach its best performance when $M = 6$, but the minimum station number gap between $M = 5$ and the optimal case is smaller than for 7-qubit codes. In low gate error case, setting $M = 5$ provides near-optimum performance while reducing the consumption of Bell pair resources. This also indicates that a larger code can reach optimal performance with a lower purification level than a smaller code.

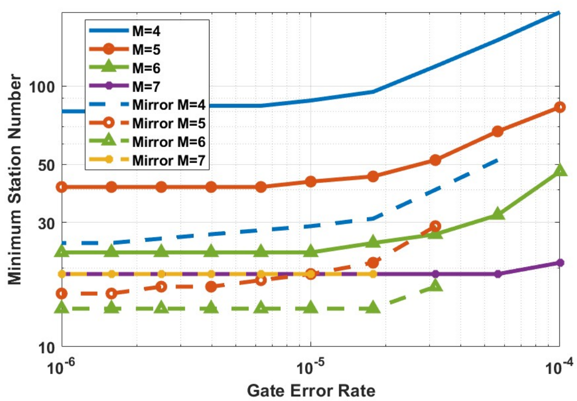

Fig. 10. Minimum intermediate station number vs local gate error rate for overall distance 500 km. Distance-3 surface code compared to mirrored CSS 9-qubit code with same parameter as in Fig.9. The parameters M denotes the purification level of the Bell states.

VII. Conclusion

We have investigated a distance-related error model for 2G quantum repeaters, which has both correlated and biased errors. Based on this error model, we showed that non-symmetric CSS codes outperform symmetric ones for 2G QRs. To improve the error correction performance, we proposed to use different codes at the neighboring stations, and introduced the class of code pairs with a mirrored structure. Non-symmetric codes target correlated errors while the mirrored structure works against biased errors. We studied the transversality of the non-local CNOT gates essential for 2G QR in a situation with two different CSS codes used for error correction at neighboring stations. We found less restrictive constraints in this case than when using the same CSS code in the neighboring stations. Also, we found sufficient conditions for having a non-local transversal CZ gate between the neighboring stations. Finally, we computed the rate of creating end-to-end entanglement in 2G QR system with limited quantum memory. Based on this, we numerically minimized the number of intermediate stations for achieving reliable end-to-end quantum communication with a given overall distance and gate error rate.

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