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A posteriori error analysis for Kirchhoff plate elements

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Outline

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Kirchhoff plate bending model

Displacement formulation. Find the deflection w such that, in the domain $\Omega \subset \mathbb{R}^2$, it holds

$$\frac{1}{6(1-\nu)} \Delta^2 w = f\,.$$

Mixed formulation. Find the deflection w, rotation $\boldsymbol{\beta}$ and the shear stress $\boldsymbol{q}$ such that it holds

$$\begin{aligned} -\operatorname{div} \boldsymbol{q} &= f\,, \\ \mathbf{div}\, \boldsymbol{m}(\boldsymbol{\beta}) + \boldsymbol{q} &= \mathbf{0}\,, \quad \text{with} \quad \boldsymbol{m}(\boldsymbol{\beta}) = \frac{1}{6}\{\boldsymbol{\varepsilon}(\boldsymbol{\beta}) + \frac{\nu}{1-\nu} \operatorname{div} \boldsymbol{\beta} \boldsymbol{I}\}\,, \\ \nabla w - \boldsymbol{\beta} &= \mathbf{0}\,. \end{aligned}$$

- Furthermore, the boundary conditions on the clamped, simply supported and free boundaries Γ_{C}, Γ_{S} and Γ_{F} are imposed.

FE formulations — Morley element

- We define the discrete space for the deflection as follows:

$$W_h = \left\{ v \in M_{2,h} \;\middle|\; \int_E [\![\frac{\partial v}{\partial \boldsymbol{n}_E}]\!] = 0 \quad \forall E \in \mathcal{E}_h \right\},$$

where E represents an edge of a triangle K in a triangulation $\mathcal{T}_h$, and $M_{2,h}$ denotes the space of the second order piecewise polynomial functions on $\mathcal{T}_h$ which are

— continuous at the vertices of all the internal triangles and

— zero at all the triangle vertices on the clamped boundary.

Finite element method. *Find* $w_h \in W_h$ *such that*

$$\sum_{K \in \mathcal{T}_h} \left(\boldsymbol{E}\boldsymbol{\varepsilon}(\nabla w_h), \boldsymbol{\varepsilon}(\nabla v) \right)_K = (f, v) \quad \forall v \in W_h \,.$$

Stabilized C^0-element

- Given an integer $k \geq 1$, we define the discrete spaces for the deflection and the rotation, respectively, as

$$W_h = \{v \in W \mid v_{|K} \in P_{k+1}(K) \ \ \forall K \in \mathcal{T}_h\},$$
$$\boldsymbol{V}_h = \{\boldsymbol{\eta} \in \boldsymbol{V} \mid \boldsymbol{\eta}_{|K} \in [P_k(K)]^2 \ \ \forall K \in \mathcal{T}_h\},$$

  where $P_k(K)$ denotes the polynomial space of degree k on K.

Finite element method. *Find* $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h$ *such that*

$$\mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = (f, v) \qquad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h\,,$$

where the bilinear form $\mathcal{A}_h$ *we split as*

$$\mathcal{A}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = \mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) + \mathcal{D}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta})\,,$$

with the stabilized (α) bending part (R–M with the limit $t \to 0$)

$$\mathcal{B}_h(z,\boldsymbol{\phi};v,\boldsymbol{\eta}) = (\boldsymbol{m}(\boldsymbol{\phi}),\boldsymbol{\varepsilon}(\boldsymbol{\eta})) - \sum_{K\in\mathcal{T}_h} \alpha h_K^2(\boldsymbol{L}\boldsymbol{\phi},\boldsymbol{L}\boldsymbol{\eta})_K$$
$$+ \sum_{K\in\mathcal{T}_h} \frac{1}{\alpha h_K^2}(\nabla z - \boldsymbol{\phi} - \alpha h_K^2\boldsymbol{L}\boldsymbol{\phi}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K$$

and the stabilized (γ) free boundary part

$$\mathcal{D}_h(z,\boldsymbol{\phi};v,\boldsymbol{\eta}) = \langle m_{ns}(\boldsymbol{\phi}), [\nabla v - \boldsymbol{\eta}]\cdot\boldsymbol{s}\rangle_{\Gamma_{\mathrm{F}}} + \langle [\nabla z - \boldsymbol{\phi}]\cdot\boldsymbol{s}, m_{ns}(\boldsymbol{\eta})\rangle_{\Gamma_{\mathrm{F}}}$$
$$+ \sum_{E\in\mathcal{F}_h} \frac{\gamma}{h_E}\langle [\nabla z - \boldsymbol{\phi}]\cdot\boldsymbol{s}, [\nabla v - \boldsymbol{\eta}]\cdot\boldsymbol{s}\rangle_E$$

for all $(z,\boldsymbol{\phi}) \in W_h \times \boldsymbol{V}_h,\ (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h,$ where $\mathcal{F}_h$ represents the collection of all the boundary edges on the free boundary Γ_{F}, and the twisting moment is $m_{ns} = \boldsymbol{s}\cdot\boldsymbol{m}\boldsymbol{n}$.

- The first term in $\mathcal{D}_h$ is for consistency, the second one for symmetry and the last one for stability.

A posteriori error estimates

- We use the following notation: $[\![\cdot]\!]$ for jumps, h_E and h_K for the edge length and the element diameter.

Interior error indicators

- For the local error indicator η_K we define: for all the elements K in the mesh $\mathcal{T}_h$, and for all the internal edges $E \in \mathcal{I}_h$,

$$\text{(Morley)} \quad \tilde{\eta}_K^2 := h_K^4 \|f\|_{0,K}^2 \,,$$

$$\text{(Stabil.)} \quad \tilde{\eta}_K^2 := h_K^4 \|f + \operatorname{div} \boldsymbol{q}_h\|_{0,K}^2 + h_K^{-2} \|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2 \,,$$

$$\text{(Morley)} \quad \eta_E^2 := h_E^{-3} \| \, [\![w_h]\!] \, \|_{0,E}^2 + h_E^{-1} \| \, [\![\frac{\partial w_h}{\partial \boldsymbol{n}_E}]\!] \|_{0,E}^2 \,,$$

$$\text{(Stabil.)} \quad \eta_E^2 := h_E^3 \| \, [\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!] \|_{0,E}^2 + h_E \| \, [\![\boldsymbol{m}(\boldsymbol{\beta}_h) \boldsymbol{n}]\!] \|_{0,E}^2 \,.$$

Boundary error indicators

- Let the boundary $\partial\Omega$ of the plate be divided into the parts of the different boundary conditions: clamped, simply supported and free, i.e., $\partial\Omega = \Gamma_{\mathrm{C}} \cup \Gamma_{\mathrm{S}} \cup \Gamma_{\mathrm{F}}$.
- For the Morley element, we assume that $\partial\Omega = \Gamma_{\mathrm{C}}$ and for the edges on the clamped boundary Γ_{C}

$$\text{(Morley)} \quad \eta_{E,C}^2 = h_E^{-3} \| [\![w_h]\!] \|_{0,E}^2 + h_E^{-1} \| [\![\frac{\partial w_h}{\partial \boldsymbol{n}_E}]\!] \|_{0,E}^2 .$$

- For the stabilized C^0-element, for the edges on the simply supported boundary Γ_{S}

$$\text{(Stabil.)} \quad \eta_{E,S}^2 := h_E \| m_{nn}(\boldsymbol{\beta}_h) \|_{0,E}^2 ,$$

and for the edges on the free boundary Γ_{F}

$$\text{(Stabil.)} \quad \eta_{E,F}^2 := h_E \| m_{nn}(\boldsymbol{\beta}_h) \|_{0,E}^2 + h_E^3 \| \frac{\partial}{\partial s} m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h \cdot \boldsymbol{n} \|_{0,E}^2 .$$

Error indicators — local and global

- Now, for any element $K \in \mathcal{T}_h$, let the local error indicator be

$$\eta_K := \Big(\tilde{\eta}_K^2 + \frac{1}{2} \sum_{\substack{E \in \mathcal{I}_h \\ E \subset \partial K}} \eta_E^2 + \sum_{\substack{E \in \mathcal{C}_h \\ E \subset \partial K}} \eta_{E,C}^2 + \sum_{\substack{E \in \mathcal{S}_h \\ E \subset \partial K}} \eta_{E,S}^2 + \sum_{\substack{E \in \mathcal{F}_h \\ E \subset \partial K}} \eta_{E,F}^2 \Big)^{1/2},$$

  with the notation

  — $\mathcal{I}_h$ for the collection of all the internal edges,

  — $\mathcal{C}_h$, $\mathcal{S}_h$ and $\mathcal{F}_h$ for the collections of all the boundary edges on Γ_{C}, Γ_{S} and Γ_{F}, respectively.

- Finally, the global error indicator is defined as

$$\eta := \Big(\sum_{K \in \mathcal{T}_h} \eta_K^2 \Big)^{1/2}.$$

Upper bounds — Reliability

- With $\mathcal{E}_h$ denoting the collection of all the triangle edges, we define the mesh dependent norms for the Morley element and for the stabilized C^0-element, respectively, as

$$|||v|||_h^2 := \sum_{K\in\mathcal{T}_h} |v|_{2,K}^2 + \sum_{E\in\mathcal{E}_h} h_E^{-3} \| [\![v]\!] \|_{0,E}^2 + \sum_{E\in\mathcal{E}_h} h_E^{-1} \| [\![\frac{\partial v}{\partial \boldsymbol{n}_E}]\!] \|_{0,E}^2 ,$$

$$|||(v,\boldsymbol{\eta})|||_h^2 := \sum_{K\in\mathcal{T}_h} |v|_{2,K}^2 + \|v\|_1^2 + \sum_{E\in\mathcal{I}_h} h_E^{-1} \| [\![\frac{\partial v}{\partial \boldsymbol{n}_E}]\!] \|_{0,E}^2$$
$$+ \sum_{K\in\mathcal{T}_h} h_K^{-2} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 + \|\boldsymbol{\eta}\|_1^2 .$$

Theorem. *There exists positive constants C such that*

$$\text{(Morley)} \quad |||w - w_h|||_h \le C\eta ,$$

$$\text{(Stabil.)} \quad |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*} \le C\eta .$$

Lower bounds — Efficiency

- Let ω_K be the collection of all the triangles in $\mathcal{T}_h$ with a nonempty intersection with the element K.

Theorem. *There exists positive constants C such that*

$$\text{(Morley)} \quad \eta_K \leq C\big(|||w - w_h|||_{h,K} + h_K^2 \|f - f_h\|_{0,K}\big)\,,$$

$$\text{(Stabil.)} \quad \eta_K \leq C\big(|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_{h,\omega_K} + h_K^2 \|f - f_h\|_{0,\omega_K} + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*,\omega_K}\big)\,,$$

for any element $K \in \mathcal{T}_h$.

Numerical results

- We have implemented the methods in the open-source finite element software *Elmer* developed by CSC – the Finnish IT Center for Science.
- Test problems with convex rectangular domains, and with known exact solutions, we have used for investigating the effectivity index for the error estimators derived.
- Non-convex domains we have used for studying the adaptive performance and robustness of the methods.

Effectivity index

$$(\text{Morley}) \quad \iota = \frac{\eta}{|||w - w_h|||_h} \qquad (\text{Stabil.}) \quad \iota = \frac{\eta}{|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h}$$

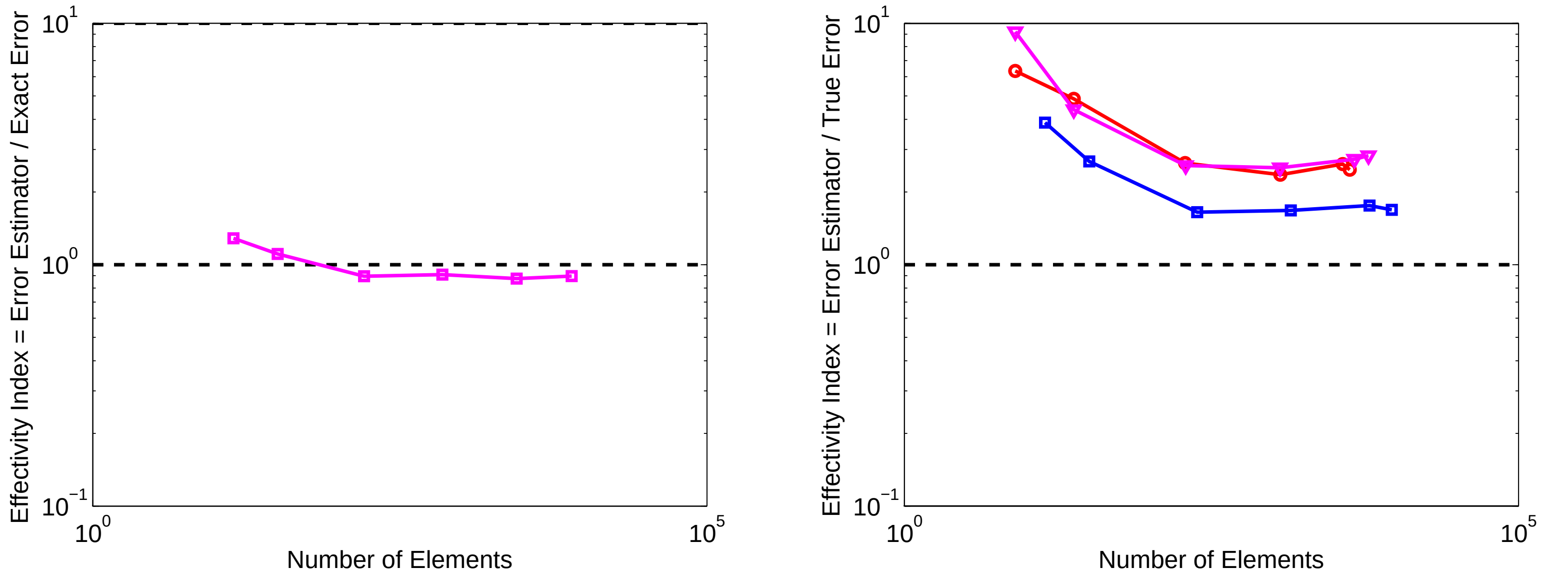


Figure 1: Effectivity index; *Left*: the Morley element (with C-boundaries); *Right*: the stabilized method (with C/S/F-boundaries).

Simply supported L-domain — Starting mesh — Deflection

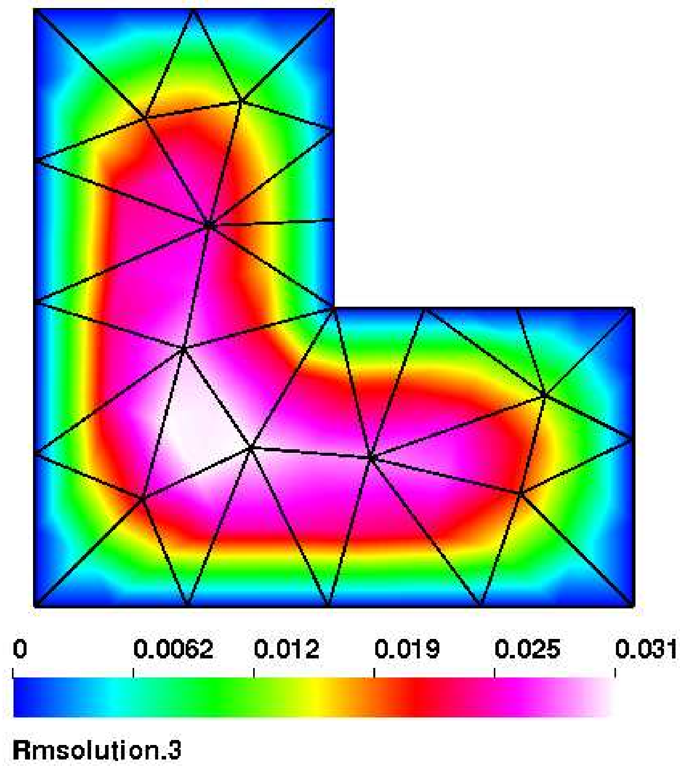


Figure 2: The stabilized method: Deflection distribution for the first mesh (constant loading).

Adaptively refined mesh — Error estimator

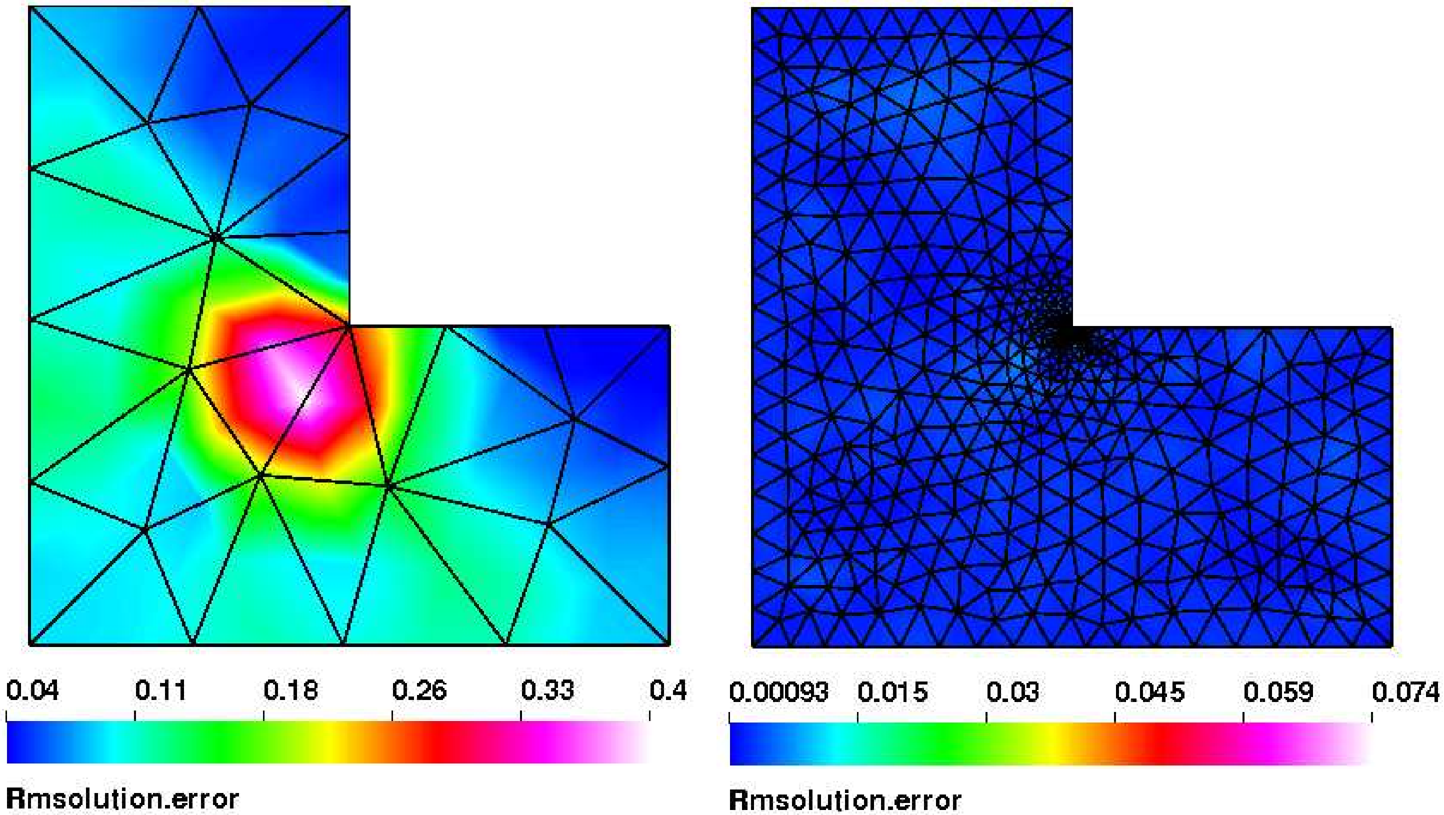


Figure 3: The stabilized method: Distribution of the error estimator for two adaptive steps.

Uniform vs. Adaptive — Convergence in the norm $||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_1 + |(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|_h$

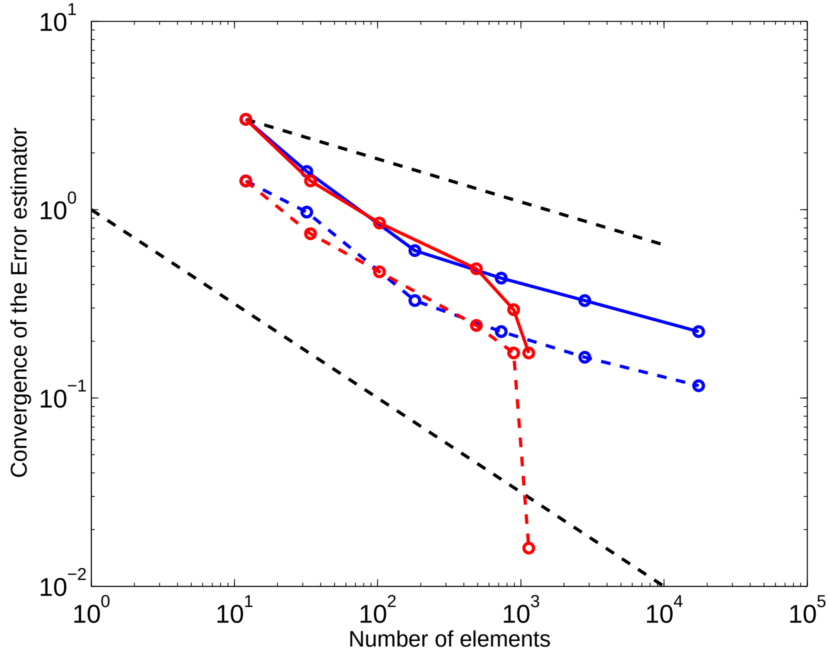


Figure 4: The stabilized method: Convergence of the error estimator for the **uniform refinements** and **adaptive refinements**; *Solid* lines for *global*, *dashed* lines for *maximum local* ones.

Clamped L-domain — Refinements — Error estimator

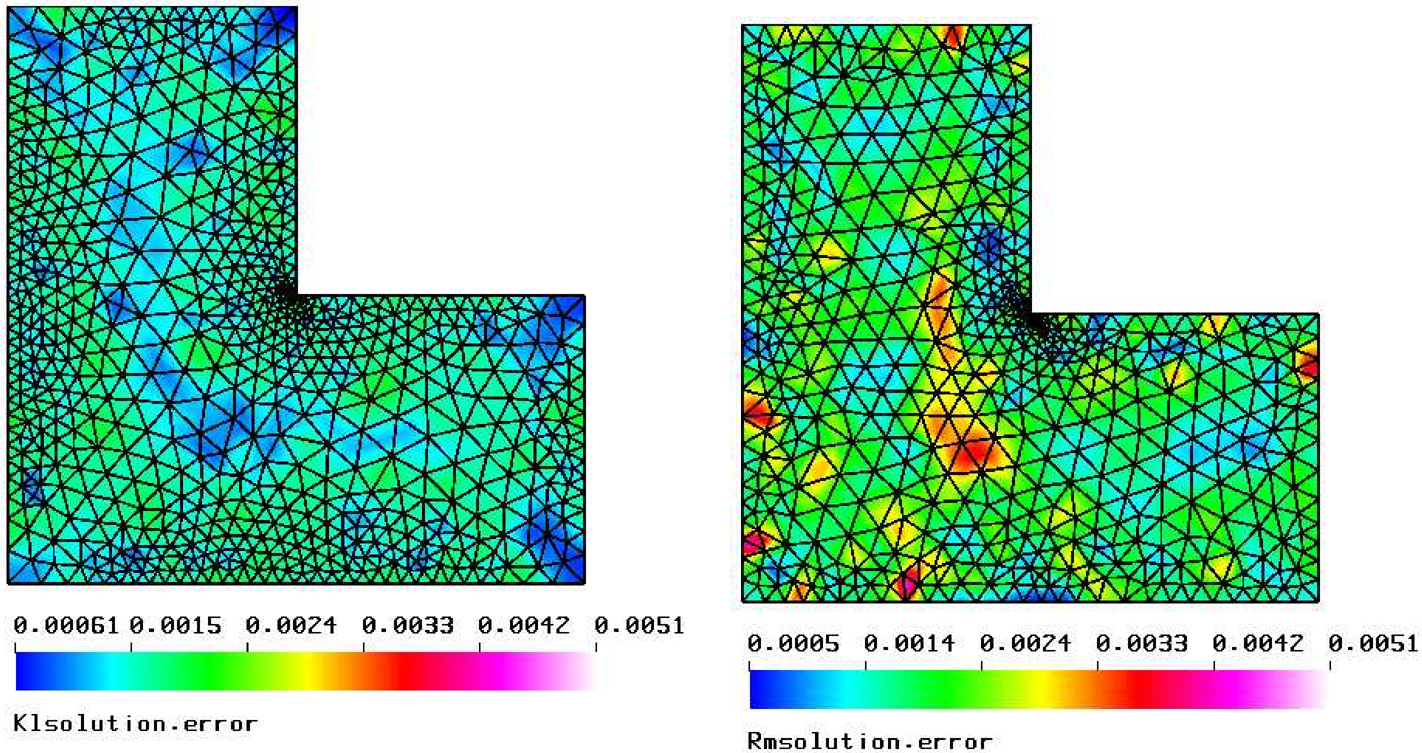


Figure 5: Distribution of the error estimator after adaptive refinements: *Left*: the Morley element; *Right*: the stabilized method.

Conclusions

- A posteriori error analysis has been accomplished for Kirchhoff plates:
  - — the Morley element for clamped boundaries
  - — the stabilized C^0-continuous element for general boundary conditions
  - — efficient and reliable error estimators for both methods.
- Numerical benchmarks confirm the adaptive performance and robustness of the error indicators.

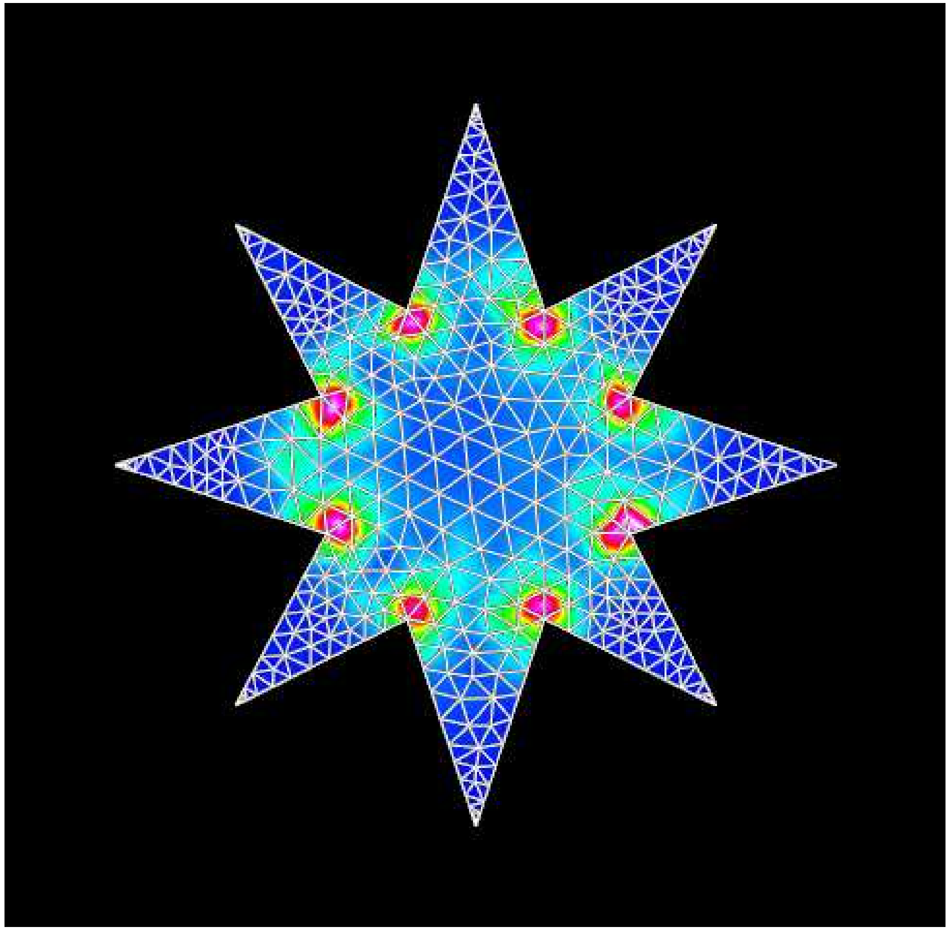
References

[1] L. Beirão da Veiga, J. Niiranen, R. Stenberg: A posteriori error estimates for the Morley plate bending element; *Numerische Matematik*, 106, 165–179 (2007).

[2] L. Beirão da Veiga, J. Niiranen, R. Stenberg: A family of C^0 finite elements for Kirchhoff plates I: Error analysis; accepted for publication in *SIAM Journal on Numerical Analysis* (2007).

[3] L. Beirão da Veiga, J. Niiranen, R. Stenberg: A family of C^0 finite elements for Kirchhoff plates II: Numerical results; accepted for publication in *Computer Methods in Applied Mechanics and Engineering* (2007).

How do we deal with a blinking star?



We snow it by adaptively refined mesh flakes!