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A posteriori error analysis for the Morley plate element

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Introduction

- Thin structures (shells, plates, membranes, beams) are the main building blocks in modern structural design.
- Beside the classical fields as civil engineering, the variety of applications have strongly increased also in many other fields as aeronautics, biomechanics, surgical medicine or microelectronics.
- In particular, new applications arise when thin structures are combined with functional, smart or composite materials (shape memory alloys, piezo-electric cheramics etc.).
- Increasing demands for accuracy and productivity have created a need for adaptive (automated, efficient, reliable) computational methods for thin structures.

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Kirchhoff plate bending model

- We consider bending of a thin planar structure occupied by

$$\mathcal{P} = \Omega \times (-\frac{t}{2}, \frac{t}{2}),$$

  where $\Omega \subset \mathbb{R}^2$ denotes the midsurface of the plate and $t \ll \operatorname{diam}(\Omega)$ denotes the thickness of the plate.

- Kinematical assumptions for the dimension reduction:
  — Straight fibres normal to the midsurface remain straight and normal.
  — Fibres normal to the midsurface do not stretch.
  — The midsurface moves only in the vertical direction.

Deformations

- Under these assumptions, with the deflection w, the displacement field $\boldsymbol{u} = (u_x, u_y, u_z)$ takes the form

$$u_x = -z\frac{\partial w(x,y)}{\partial x}\,, \quad u_y = -z\frac{\partial w(x,y)}{\partial y}\,, \quad u_z = w(x,y)\,.$$

- The corresponding deformation is defined by the symmetric linear strain tensor

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right)\,,$$

in the component form as

$$e_{xx} = -z\frac{\partial^2 w}{\partial x^2}\,, \quad e_{yy} = -z\frac{\partial^2 w}{\partial y^2}\,, \quad e_{zz} = 0\,,$$

$$e_{xy} = -z\frac{\partial^2 w}{\partial x \partial y}\,, \quad e_{xz} = 0\,, \quad e_{yz} = 0\,.$$

Stress resultants

Next, we define the stress resultants, the moments and the shear forces:

$$\boldsymbol{M} = \begin{pmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} \quad \text{with} \quad M_{ij} = -\int_{-t/2}^{t/2} z\,\sigma_{ij}\,dz\,, \quad i,j = x,y\,,$$

$$\boldsymbol{Q} = \begin{pmatrix} Q_x \\ Q_y \end{pmatrix} \quad \text{with} \quad Q_i = \int_{-t/2}^{t/2} \sigma_{iz}\,dz\,, \quad i = x,y\,,$$

where the stress tensor is assumed to be symmetric:

$$\sigma_{ij} = \sigma_{ji}\,, \quad i,j = x,y,z.$$

Equilibrium equations and boundary conditions

The principle of virtual work gives, with the load resultant F, the equilibrium equation

$$\operatorname{div} \mathbf{div}\, \boldsymbol{M} = F \quad \text{with} \quad \mathbf{div}\, \boldsymbol{M} + \boldsymbol{Q} = \boldsymbol{0}\,.$$

and the boundary conditions

$$\begin{aligned}
&w = 0\,, \quad \nabla w \cdot \boldsymbol{n} = 0 && \text{on } \Gamma_{\mathrm{C}}\,,\\
&w = 0\,, \quad \boldsymbol{n} \cdot \boldsymbol{M}\boldsymbol{n} = 0 && \text{on } \Gamma_{\mathrm{S}}\,,\\
&\boldsymbol{n} \cdot \boldsymbol{M}\boldsymbol{n} = 0\,, \quad \tfrac{\partial^2}{\partial \boldsymbol{s}^2}(\boldsymbol{s} \cdot \boldsymbol{M}\boldsymbol{n}) + \boldsymbol{n} \cdot \mathbf{div}\, \boldsymbol{M} = 0 && \text{on } \Gamma_{\mathrm{F}}\,,\\
&(\boldsymbol{s}_1 \cdot \boldsymbol{M}\boldsymbol{n}_1)(c) = (\boldsymbol{s}_2 \cdot \boldsymbol{M}\boldsymbol{n}_2)(c) && \forall c \in \mathcal{V}\,,
\end{aligned}$$

where the indices 1 and 2 refer to the sides of the boundary angle at a corner point c on the free boundary Γ_{F}.

Constitutive assumptions

- The material of the plate is assumed to be
  - — linearly elastic (defined by the generalized Hooke's law)
  - — homogeneous (independent of the coordinates x, y, z)
  - — isotropic (independent of the coordinate system).
- Furthermore, we assume that the transverse normal stress vanishes: $\sigma_{zz} = 0$.

Variational formulation

Let the deflection w belong to the Sobolev space

$$W = \{v \in H^2(\Omega) \mid v = 0 \text{ on } \Gamma_C \cup \Gamma_S,\ \nabla v \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_C\},$$

where $\boldsymbol{n}$ indicates the unit outward normal to the boundary Γ.

Problem. *Variational formulation: Find $w \in W$ such that*

$$(\boldsymbol{E}\boldsymbol{\varepsilon}(\nabla w), \boldsymbol{\varepsilon}(\nabla v)) = (f, v) \quad \forall v \in W,$$

with the elasticity tensor $\boldsymbol{E}$ defined as

$$\boldsymbol{E}\boldsymbol{\varepsilon} = \frac{\mathsf{E}}{12(1+\nu)}\Big(\boldsymbol{\varepsilon} + \frac{\nu}{1-\nu}\operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}\Big) \quad \forall \boldsymbol{\varepsilon} \in \mathbb{R}^{2\times 2},$$

with Young's modulus E and the Poisson ratio ν.

Morley finite element formulation

Let E denote an edge of a triangle K in a triangulation $\mathcal{T}_h$. We define the discrete space for the deflection as

$$W_h = \left\{ v \in M_{2,h} \;\middle|\; \int_E [\![\nabla v \cdot \boldsymbol{n}_E]\!] = 0 \quad \forall E \in \mathcal{E}_h^i \cup \mathcal{E}_h^c \right\},$$

where $M_{2,h}$ denotes the space of the second order piecewise polynomial functions on $\mathcal{T}_h$ which are

— continuous at the vertices of all the internal triangles and

— zero at all the triangle vertices of $\Gamma_C \cup \Gamma_S$.

Finite element method. *Morley: Find $w_h \in W_h$ such that*

$$\sum_{K \in \mathcal{T}_h} \left(\boldsymbol{E}\boldsymbol{\varepsilon}(\nabla w_h), \boldsymbol{\varepsilon}(\nabla v) \right)_K = (f, v) \quad \forall v \in W_h \,.$$

A priori error estimate

The method is stable and convergent with respect to the following discrete norm on the space $W_h + H^2$:

$$|||v|||_h^2 := \sum_{K \in \mathcal{T}_h} |v|_{2,K}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-3} \| [\![v]\!] \|_{0,E}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-1} \| [\![\frac{\partial v}{\partial \boldsymbol{n}_E}]\!] \|_{0,E}^2 \,,$$

Proposition. *(Shi 90, Ming and Xu 06) Assuming that* $\Gamma = \Gamma_C$ *there exists a positive constant* C *such that*

$$|||w - w_h|||_h \le C h \left(|w|_{H^3(\Omega)} + h \|f\|_{L^2(\Omega)} \right) \,.$$

The numerical results indicate the same convergence rate for general boundary conditions as well.

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A posteriori error estimates

- We use the following notation: $[\![\cdot]\!]$ for jumps (and traces), h_E and h_K for the edge length and the element diameter.

Interior error indicators

- For all the elements K in the mesh $\mathcal{T}_h$,

$$\tilde{\eta}_K^2 := h_K^4 \| f \|_{0,K}^2 \,,$$

and for all the internal edges $E \in \mathcal{I}_h$,

$$\eta_E^2 := h_E^{-3} \| \, [\![w_h]\!] \, \|_{0,E}^2 + h_E^{-1} \| \, [\![\frac{\partial w_h}{\partial \boldsymbol{n}_E}]\!] \|_{0,E}^2 \,.$$

Boundary error indicators

- The boundary of the plate is divided into clamped, simply supported and free parts:

$$\Gamma := \partial\Omega = \Gamma_C \cup \Gamma_S \cup \Gamma_F \,.$$

- For edges on the clamped and simply supported boundaries Γ_{C} and boundary Γ_{S}, respectively,

$$\eta_{E,C}^2 := h_E^{-3} \| \,[\![w_h]\!]\, \|_{0,E}^2 + h_E^{-1} \| \,[\![\frac{\partial w_h}{\partial \boldsymbol{n}_E}]\!]\, \|_{0,E}^2,$$

$$\eta_{E,S}^2 := h_E^{-3} \| \,[\![w_h]\!]\, \|_{0,E}^2.$$

Error indicators — local and global

- For any element $K \in \mathcal{T}_h$, let the local error indicator be

$$\eta_K := \Big(\tilde{\eta}_K^2 + \frac{1}{2} \sum_{\substack{E \in \mathcal{I}_h \\ E \subset \partial K}} \eta_E^2 + \sum_{\substack{E \in \mathcal{C}_h \\ E \subset \partial K}} \eta_{E,C}^2 + \sum_{\substack{E \in \mathcal{S}_h \\ E \subset \partial K}} \eta_{E,S}^2 \Big)^{1/2},$$

with the notation

— $\mathcal{I}_h$ for the collection of all the internal edges,

— $\mathcal{C}_h$ and $\mathcal{S}_h$ for the collections of all the boundary edges on Γ_{C} and Γ_{S}, respectively.

- The global error indicator is defined as

$$\eta_h := \Big(\sum_{K \in \mathcal{T}_h} \eta_K^2 \Big)^{1/2}.$$

Upper bound — Reliability

Theorem. *Reliability: There exists a positive constant C such that*

$$|||w - w_h|||_h \leq C\eta_h \,.$$

Lower bound — Efficiency

Theorem. *Efficiency: For any element K, there exists a positive constant C_K such that*

$$\eta_K \leq C_K \big(|||w - w_h|||_{h,K} + h_K^2 \|f - f_h\|_{0,K} \big) \,.$$

Efficiency is proved by standard arguments; reliability needs a new Clément-type interpolant and a new Helmholtz-type decomposition.

Techniques for the analysis — Helmholtz decomposition

Lemma. *Let $\boldsymbol{\sigma}$ be a second order tensor field in $L^2(\Omega; \mathbb{R}^{2\times 2})$. Then, there exist $\psi \in W$, $\rho \in L^2_0(\Omega)$ and $\boldsymbol{\phi} \in [\tilde{H}^1(\Omega)]^2$ such that*

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon}(\nabla\psi) + \boldsymbol{\rho} + \mathbf{Curl}\,\boldsymbol{\phi}, \quad \text{with} \quad \boldsymbol{\rho} = \begin{pmatrix} 0 & -\rho \\ \rho & 0 \end{pmatrix}.$$

$$\|\psi\|_{H^2(\Omega)} + \|\rho\|_{L^2(\Omega)} + \|\boldsymbol{\phi}\|_{H^1(\Omega)} \leq C\|\boldsymbol{\sigma}\|_{L^2(\Omega)}.$$

Here $\tilde{H}^m(\Omega)$, $m \in \mathbb{N}$, indicate the quotient space of $H^m(\Omega)$ where the seminorm $|\cdot|_{H^m(\Omega)}$ is null.

In analysis, Lemma is applied to the tensor field $\boldsymbol{E}\boldsymbol{\varepsilon}(\nabla(w - w_h))$.

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Numerical results

- We have implemented the method in the open-source finite element software *Elmer* developed by CSC – the Finnish IT Center for Science.
- The software provides error balancing strategy and complete remeshing for triangular meshes.
- We have used test problems with convex rectangular domains – and with known exact solutions – for investigating the effectivity index for the error estimator derived.
- Non-convex domains we have used for studying the adaptive performance and robustness of the method.

Effectivity index $\iota = \frac{\eta_h}{|||w - w_h|||_h}$

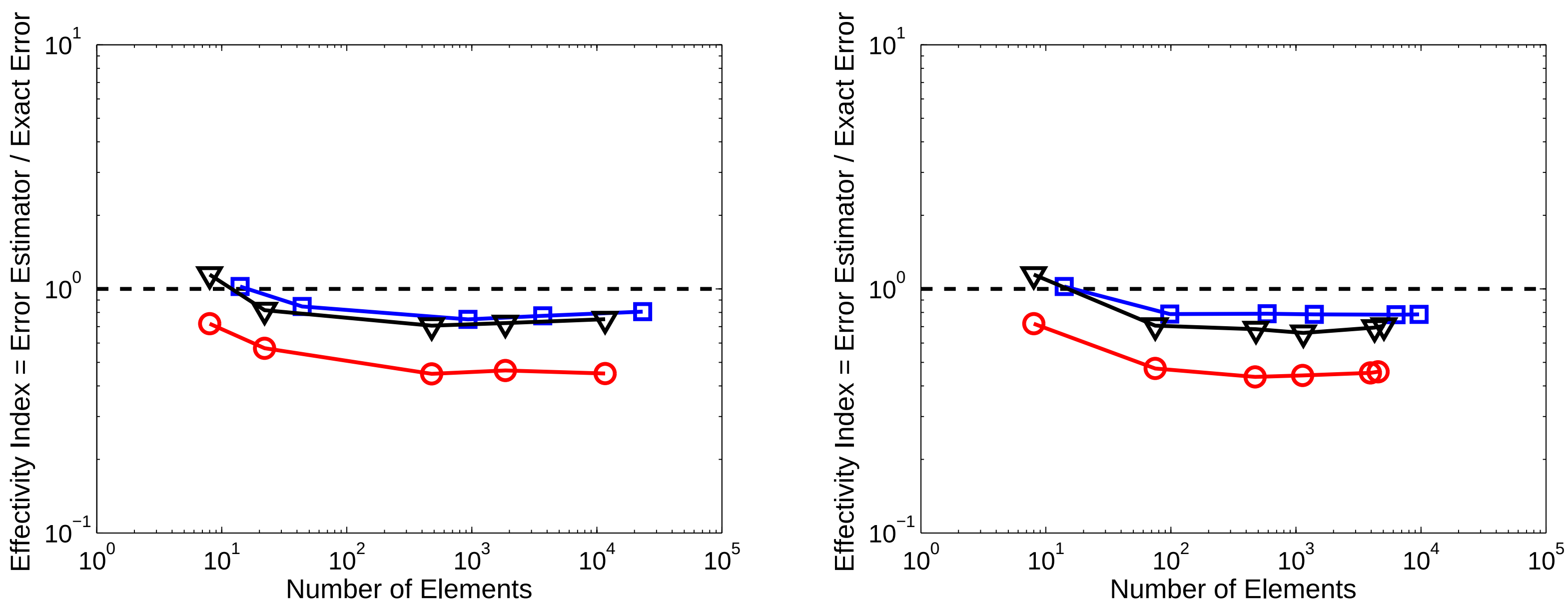

Figure 1: *Left*: uniform refinements; *Right*: adaptive refinements. Clamped (squares), simply supported (circles) and free (triangles) boundaries included.

Adaptively refined mesh — Error estimator
Simply supported L-corner

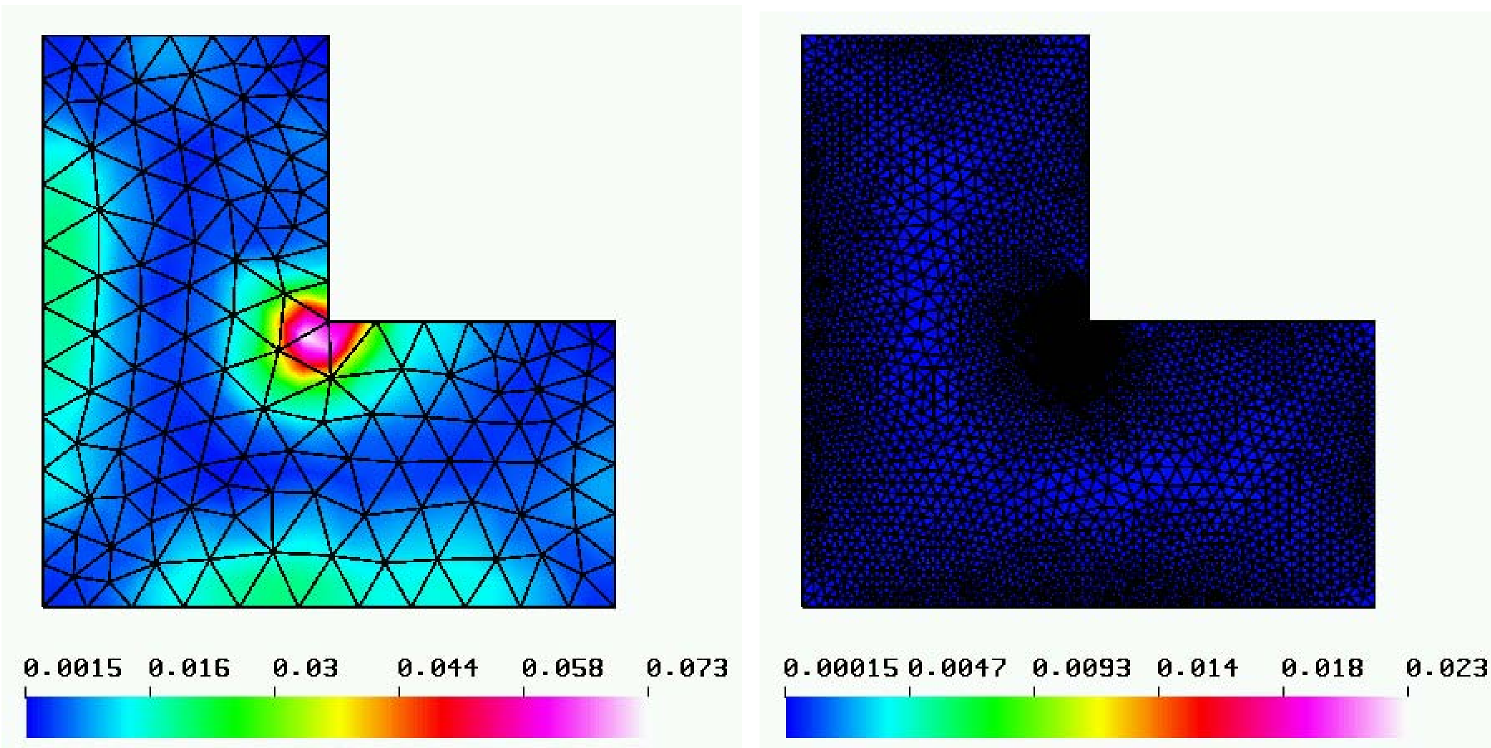

Figure 2: Simply supported L-shaped domain: Distribution of the error estimator for two adaptive steps.

Uniform vs. Adaptive — Convergence

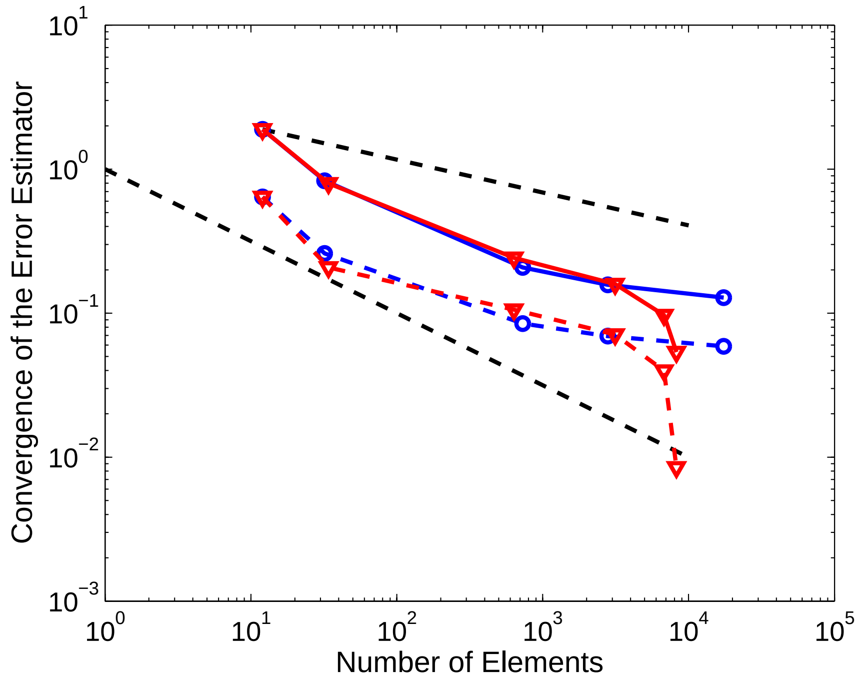

Figure 3: Simply supported L-shaped domain: Convergence of the error estimator for the **uniform refinements** and **adaptive refinements**; *Solid* lines for *global*, *dashed* lines for *maximum local* ones.

Adaptively refined mesh — Error estimator
Clamped L-corner

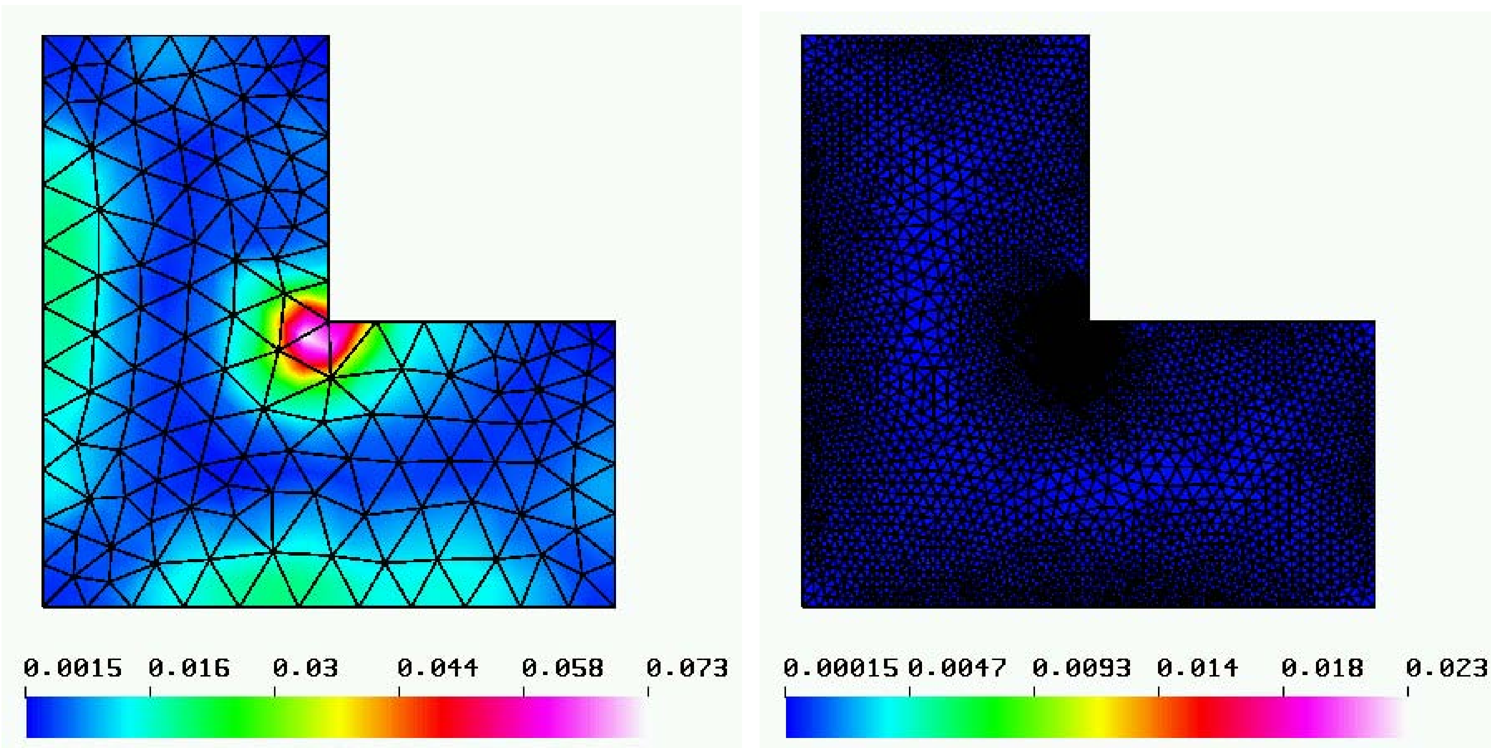

Figure 4: Simply supported L-shaped domain with a clamped L-corner: Distribution of the error estimator for two adaptive steps.

Uniform vs. Adaptive — Convergence

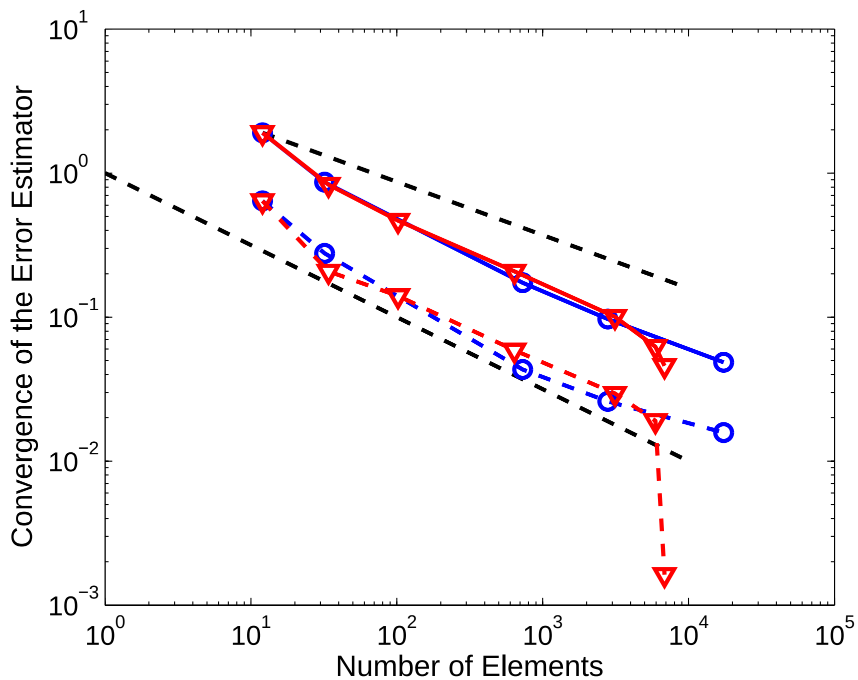

Figure 5: Clamped L-corner: Convergence of the error estimator for the **uniform refinements** and **adaptive refinements**; *Solid* lines for *global*, *dashed* lines for *maximum local* ones.

Adaptively refined mesh — Error estimator
Simply supported M-domain

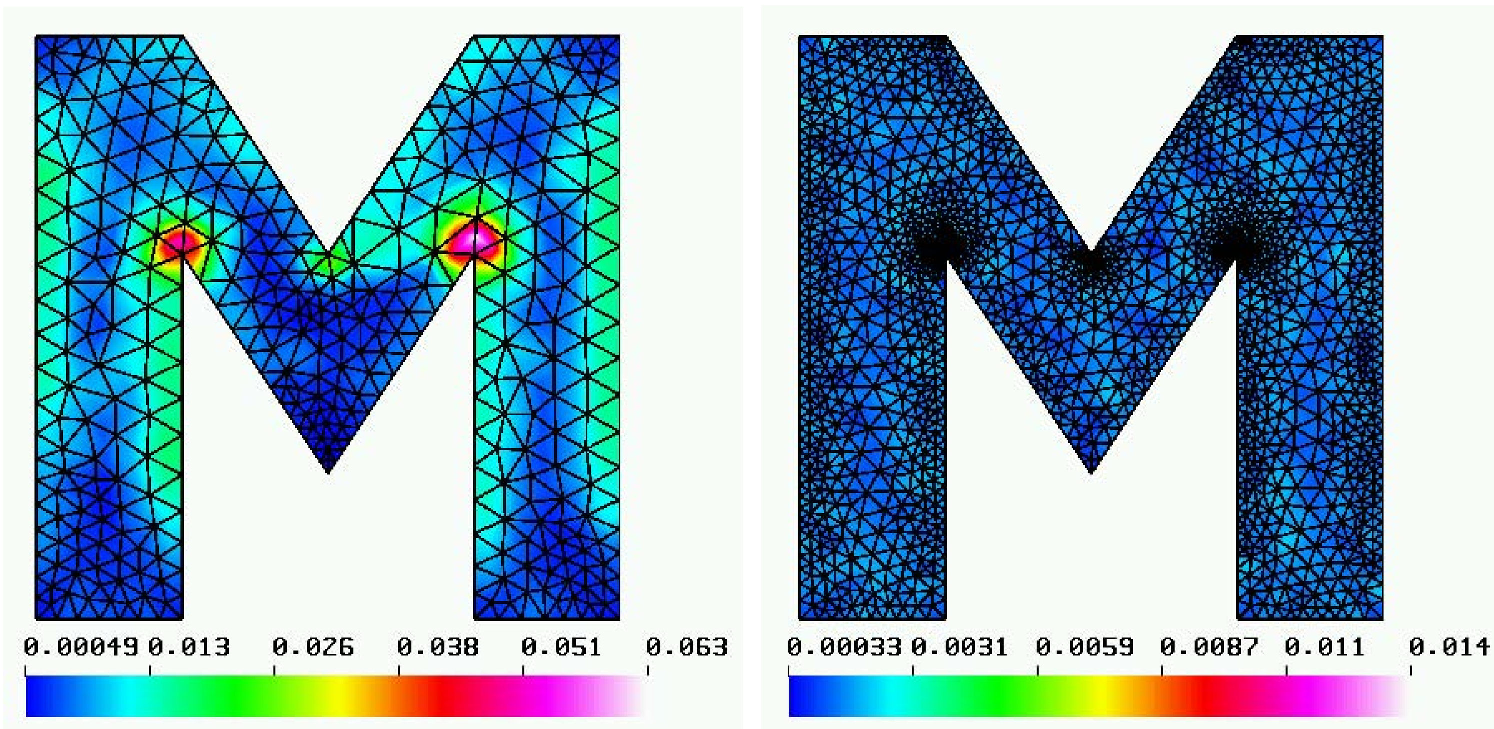

Figure 6: Simply supported M-shaped domain: Distribution of the error estimator for two adaptive steps.

Uniform vs. Adaptive — Convergence

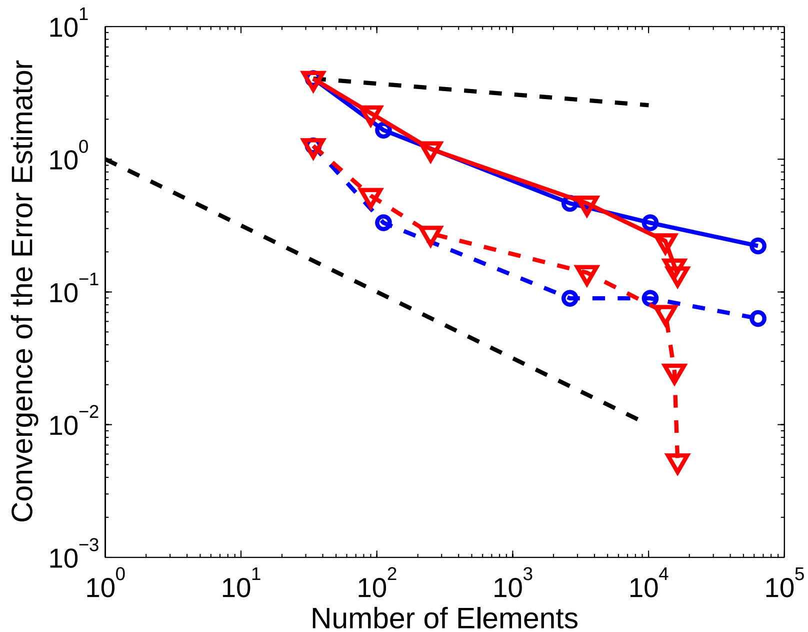

Figure 7: Simply supported M-shaped domain: Convergence of the error estimator for the **uniform refinements** and **adaptive refinements**; *Solid* lines for *global*, *dashed* lines for *maximum local* ones.

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Conclusions and Discussion

Advantages

- Reliability: computable (non-guaranteed due to C) global upper bound for the error.
- Efficiency: computable (non-guaranteed due to C_K) local lower bound.
- Robustness: C_K independent of the mesh size, data and the solution.
- Computational costs: small (local) compared to solving the problem itself.

Disadvantages

- Residual based error estimates in the energy norm only
  — no estimates for other quantities of interest.

- Method dependent: applicape for the Morley element only
  — although the techniques can be generalized.

- Valid only for static problem with transversal loading and isotropic, homogeneous, linearly elastic material
  — so far.

References

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