Computational results for the superconvergence and postprocessing of MITC plate elements

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Summary We summarize the main parts of the theoretical results introduced and analyzed in [5] for the MITC plate elements [2], [4]. We also illustrate and verify the superconvergence properties and the post-processing method with various numerical computations.

Introduction

The deflection approximation of the MITC plate elements [2], [4] is shown to be superconvergent with respect to a special interpolation operator [5]. This property holds in the H^1-norm and the interpolation operator is closely related to the reduction operator used in the MITC methods. A part of the superconvergence result is, roughly speaking, that the vertex values obtained with the MITC methods are superconvergent. This may be an explanation why these methods have become so popular.

By utilizing the superconvergence property a postprocessing method has been introduced in [5] — to improve the accuracy of the deflection approximation. The new approximation for the deflection is constructed element by element which implies low computational costs. The new approximation is a piecewise polynomial of one degree higher than the original one.

Here we first summarize the main parts of the theoretical results. Then we show various computational results illustrating the superconvergence properties of the original approximation and confirming the improved accuracy of the postprocessed approximation. In the numerical tests both uniform and non-uniform meshes are used and cases with different kinds of boundary conditions are studied.

MITC finite elements for Reissner-Mindlin plates

We consider a linearly elastic and isotropic plate with the shear modulus G and the Poisson ratio ν. The midsurface of the undeformed plate is $\Omega \subset \mathbb{R}^2$ and the plate thickness t is constant. The boundary of the plate we divide into hard clamped, hard simply supported and free parts: $\partial\Omega = \Gamma_{\mathrm{C}} \cup \Gamma_{\mathrm{SS}} \cup \Gamma_{\mathrm{F}}$. The spaces of kinematically admissible deflections and rotations are then

$$W = \{v \in H^1(\Omega) \ \mid \ v_{|\Gamma_{\mathrm{C}}} = 0, \ v_{|\Gamma_{\mathrm{SS}}} = 0\}, \tag{1}$$

$$\boldsymbol{V} = \{\boldsymbol{\eta} \in [H^1(\Omega)]^2 \ \mid \ \boldsymbol{\eta}_{|\Gamma_{\mathrm{C}}} = \mathbf{0}, \ (\boldsymbol{\eta} \cdot \boldsymbol{\tau})_{|\Gamma_{\mathrm{SS}}} = 0\}, \tag{2}$$

where $\boldsymbol{\tau}$ is the unit tangent to the boundary. For the analysis the problem is written in mixed form in which the shear force $\boldsymbol{q} = t^{-2}(\nabla w - \boldsymbol{\beta})$ is taken as an independent unknown in the space $\boldsymbol{Q} = [L^2(\Omega)]^2$ [4], [5]. For the bilinear form we define the bending part and the linear strain tensor:

$$a(\boldsymbol{\phi},\boldsymbol{\eta}) = \frac{1}{6}\{(\boldsymbol{\varepsilon}(\boldsymbol{\phi}),\boldsymbol{\varepsilon}(\boldsymbol{\eta})) + \frac{\nu}{1-\nu}(\operatorname{div}\boldsymbol{\phi},\operatorname{div}\boldsymbol{\eta})\}, \tag{3}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{\eta}) = \frac{1}{2}(\nabla\boldsymbol{\eta} + (\nabla\boldsymbol{\eta})^T). \tag{4}$$

We consider the triangular family but we emphasize that all the results are valid for quadrilateral families as well. By $\mathcal{C}_h$ we denote the triangulation of $\overline{\Omega}$. As usual, we denote $h = \max_{K\in\mathcal{C}_h} h_K$, where h_K is the diameter of K. The space of polynomials of degree k on K is denoted by $P_k(K)$. By C we denote positive constants independent of the thickness t and the mesh size h.
In the MITC methods [2], [4] the finite element subspaces $W_h \subset W$ and $\boldsymbol{V}_h \subset \boldsymbol{V}$ are defined for the polynomial degree $k \geq 2$ as

$$W_h = \{w \in W \ \mid \ w_{|K} \in P_k(K) \ \ \forall K \in \mathcal{C}_h\}, \tag{5}$$

$$\boldsymbol{V}_h = \{\boldsymbol{\eta} \in \boldsymbol{V} \ \mid \ \boldsymbol{\eta}_{|K} \in [P_k(K)]^2 \oplus [B_{k+1}(K)]^2 \ \ \forall K \in \mathcal{C}_h\}, \tag{6}$$

with the ”bubble space”

$$B_{k+1}(K) = \{b = b_3 p \ \mid \ p \in \tilde{P}_{k-2}(K), \ b_3 \in P_3(K), \ b_{3|E} = 0 \ \ \forall E \subset \partial K\}, \tag{7}$$

where $\widetilde{P}_{k-2}(K)$ is the space of homogeneous polynomials of degree $k-2$ on the element K. The discrete shear space is the rotated Raviart-Thomas space of order $k-1$,

$$\boldsymbol{Q}_h = \{\, \boldsymbol{r} \in \boldsymbol{H}(\mathrm{rot};\Omega) \ \mid \ \boldsymbol{r}_{|K} \in [P_{k-1}(K)]^2 \oplus (y,-x)\tilde{P}_{k-1}(K) \ \forall K \in \mathcal{C}_h \,\}. \tag{8}$$

The reduction operator $\boldsymbol{R}_h : \boldsymbol{H}(\mathrm{rot};\Omega) \to \boldsymbol{Q}_h$ is defined locally, with $\boldsymbol{R}_K = \boldsymbol{R}_{h|K}$, through the conditions

$$\langle(\boldsymbol{R}_K\boldsymbol{\eta} - \boldsymbol{\eta})\cdot\boldsymbol{\tau}_E, p\rangle_E = 0 \ \ \forall p \in P_{k-1}(E) \ \ \forall E \subset \partial K, \tag{9}$$

$$(\boldsymbol{R}_K\boldsymbol{\eta} - \boldsymbol{\eta}, \boldsymbol{p})_K = 0 \ \ \forall \boldsymbol{p} \in [P_{k-2}(K)]^2, \tag{10}$$

where E denotes an edge to K and $\boldsymbol{\tau}_E$ is the unit tangent to E. $(\cdot,\cdot)_K$ and $\langle\cdot,\cdot\rangle_E$ are the L^2-inner products.
With these assumptions and notation the MITC finite element method for the Reissner-Mindlin plate model, under the transverse loading $g \in H^{-1}(\Omega)$, can be written in the following form [4], [5]: *Find the deflection $w_h \in W_h$ and the rotation $\boldsymbol{\beta}_h \in \boldsymbol{V}_h$ such that*

$$a(\boldsymbol{\beta}_h,\boldsymbol{\eta}) + \frac{1}{t^2}(\boldsymbol{R}_h(\nabla w_h - \boldsymbol{\beta}_h), \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta})) = (g,v) \ \ \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h. \tag{11}$$

The discrete shear force is $\boldsymbol{q}_h = t^{-2}\boldsymbol{R}_h(\nabla w_h - \boldsymbol{\beta}_h) \in \boldsymbol{Q}_h$.

Superconvergence and postprocessing
For the superconvergence result we need the classical quasi-optimal interpolation operator $I_h : H^s(\Omega) \to W_h$, $s > 1$, [5]: *With a vertex a and an edge E of the triangle K, we define*

$$(v - I_K v)(a) = 0 \ \ \forall a \in K, \tag{12}$$

$$\langle v - I_K v, p\rangle_E = 0 \ \ \forall p \in P_{k-2}(E) \ \ \forall E \subset K, \tag{13}$$

$$(v - I_K v, p)_K = 0 \ \ \forall p \in P_{k-3}(K), \tag{14}$$

with $I_K = I_{h|K} \ \ \forall K \in \mathcal{C}_h$. The key property for the proof of the superconvergence is the close connection between the interpolation and reduction operators [5, Lemma 4.5]:

$$\boldsymbol{R}_h \nabla v = \nabla I_h v \ \ \forall v \in H^s(\Omega), \ s \geq 2. \tag{15}$$

Then the following superconvergence result holds [5, Theorem 4.1]:

Theorem 1. *There is a positive constant C such that*

$$\|\nabla(I_h w - w_h)\|_{0,K} \leq C h_K \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_{1,K} + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_{0,K} + t^2 \|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,K} + t^2 \|\boldsymbol{q} - \boldsymbol{R}_h \boldsymbol{q}\|_{0,K}. \quad (16)$$

For one element this gives a local improvement of order $h_K + t$ when comparing the convergence rate for $\|w_h - I_h w\|_1$ to the rates for both $\|w - w_h\|_1$ and $\|w - I_h w\|_1$ [5, Theorem 3.2, Lemma 4.2]. Since $I_h w$ interpolates w at the vertices (see Eq. (12)) this also gives an indication that the vertex values of w_h converge with an improved speed.
In the postprocessing we construct an improved approximation for the deflection in the space

$$W_h^* = \{v \in W \ \mid \ v_{|K} \in P_{k+1}(K) \ \ \forall K \in \mathcal{C}_h\}. \quad (17)$$

For the postprocessing we first introduce the interpolation operator $I_h^* : H^s(\Omega) \rightarrow W_h^*$, $s > 1$, by the equations (12)—(14) with $k + 1$ in place of k. Thus, the interpolation operators I_h^* and I_h are hierarchical, and the local spaces for the additional degrees of freedom are defined as

$$\widehat{W}(K) = \{v \in P_{k+1}(K) \ \mid \ I_K v = 0, \ (v, p)_K = 0 \ \ \forall p \in \tilde{P}_{k-2}(K)\}, \quad (18)$$
$$\overline{W}(K) = \{v \in P_{k+1}(K) \ \mid \ I_K v = 0, \ \langle v, p \rangle_E = 0 \ \ \forall p \in \tilde{P}_{k-1}(E) \ \ \forall E \subset K\}. \quad (19)$$

Furthermore, the space $\boldsymbol{Q}_h^*$ follows the definition (8) and the operator $\boldsymbol{R}_h^*$ the definitions (9) and (10), with $k + 1$ in place of k. Now the method is defined as follows [5]:

Postprocessing scheme. *For all the triangles $K \in \mathcal{C}_h$ find the local postprocessed finite element deflection $w_{h|K}^* \in P_{k+1}(K) = P_k(K) \oplus \widehat{W}(K) \oplus \overline{W}(K)$ such that*

$$I_h w_{h|K}^* = w_{h|K}, \quad (20)$$
$$\langle \nabla w_h^* \cdot \boldsymbol{\tau}_E, \nabla \hat{v} \cdot \boldsymbol{\tau}_E \rangle_E = \langle (\boldsymbol{\beta}_h + t^2 \boldsymbol{q}_h) \cdot \boldsymbol{\tau}_E, \nabla \hat{v} \cdot \boldsymbol{\tau}_E \rangle_E \ \ \forall E \subset \partial K, \quad \forall \hat{v} \in \widehat{W}(K), \quad (21)$$
$$(\nabla w_h^*, \nabla \bar{v})_K = (\boldsymbol{\beta}_h + t^2 \boldsymbol{q}_h, \nabla \bar{v})_K \ \ \forall \bar{v} \in \overline{W}(K). \quad (22)$$

We note that the postprocessed deflection is conforming since $(\boldsymbol{\beta}_h + t^2 \boldsymbol{q}_h) \cdot \boldsymbol{\tau}$ is continuous along inter element boundaries. For the method we have the following error estimate [5, Theorem 5.1]:

Theorem 2. *There is a positive constant C such that*

$$\begin{aligned} &\|\nabla(w - w_h^*)\|_{0,K} \\ &\quad \leq C\big(h_K \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_{1,K} + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_{0,K} + t^2 \|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,K} \\ &\qquad + \|\nabla(w - I_h^* w)\|_{0,K} + \|\boldsymbol{\beta} - \boldsymbol{R}_h^* \boldsymbol{\beta}\|_{0,K} + t^2 \|\boldsymbol{q} - \boldsymbol{R}_h^* \boldsymbol{q}\|_{0,K} + t^2 \|\boldsymbol{q} - \boldsymbol{R}_h \boldsymbol{q}\|_{0,K}\big). \end{aligned} \quad (23)$$

Also this result is local and it is made up of two parts: The first part is related to the error of the original method and the second part consists of interpolation estimates — both parts giving an improvement by the factor $h_K + t$ compared to the original approximation.

Selected computational results

Our numerical computations are performed for a test problem for which an analytical solution has been obtained in [1]. The domain is the semi-infinite region $\Omega = \{(x, y) \in \mathbb{R}^2 \ \mid \ y > 0\}$ and the loading is $g = \frac{1}{G} \cos x$. The Poisson ratio is $\nu = 0.3$, the shear modulus is $G = 1/(2(1 + \nu))$, the shear corrector factor is $\kappa = 1$ and the thickness is $t = 0.01$. The boundary $\Gamma = \{(x, y) \in \mathbb{R}^2 \ \mid \ y = 0\}$ is either hard simply supported or free. We have used both uniform and non-uniform meshes with quadratic ($k = 2$) and cubic ($k = 3$) elements.

The numerical results are clearly in accordance with the theory: In the interior of the plate the convergence rate of the original finite element deflection in the H^1-norm is $r \approx k$, and the convergence rate of the postprocessed finite element deflection is $r^* \approx k+1 \approx r+1$, as seen in Fig. 1 (left). The behavior in the L^2-norm looks very similar, although to rigorously prove the improvement in that case seems to be difficult. In the boundary region of the free edge case the rate of convergence rapidly slows down for both the original and the postprocessed deflection, as proved in [6], [3]. But still, a significant accuracy improvement is obtained, especially for coarse meshes and lower order elements. Furthermore, the superaccuracy of the vertex values is obvious, as seen in Fig. 1 (right).

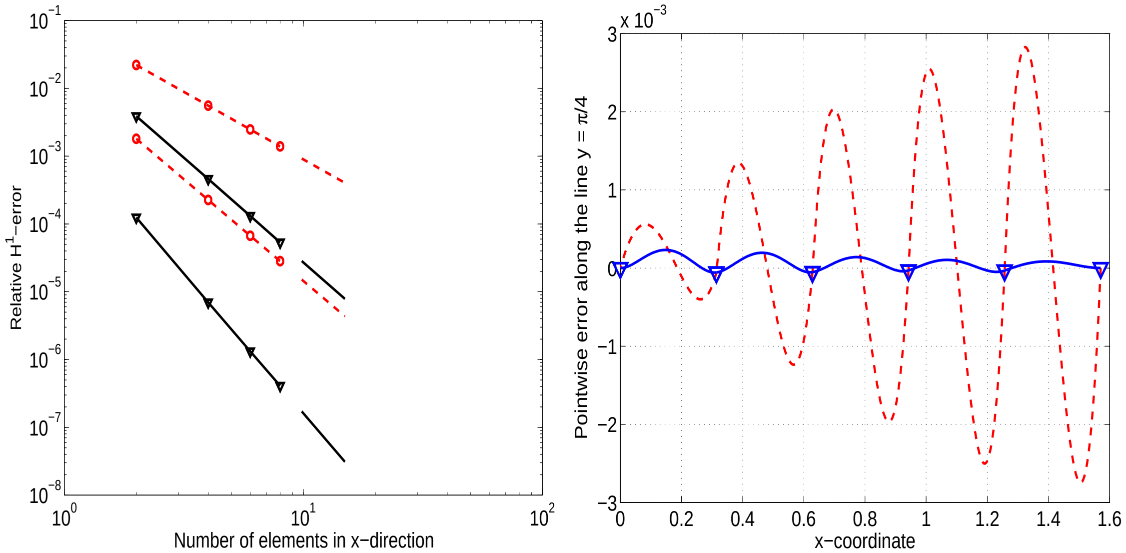

Figure 1: *Left:* Simply supported edge; Interior region; Uniform mesh; H^1- error with $k = 2, 3$ (dashed line for the original, solid line for the postprocessed deflection).
Right: Free edge; Boundary region; Non-uniform mesh; Pointwise error along the line $y = \pi/4$ with $k = 2$ (dashed line for the original, solid line for the postprocessed deflection, triangles for the vertex values).

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